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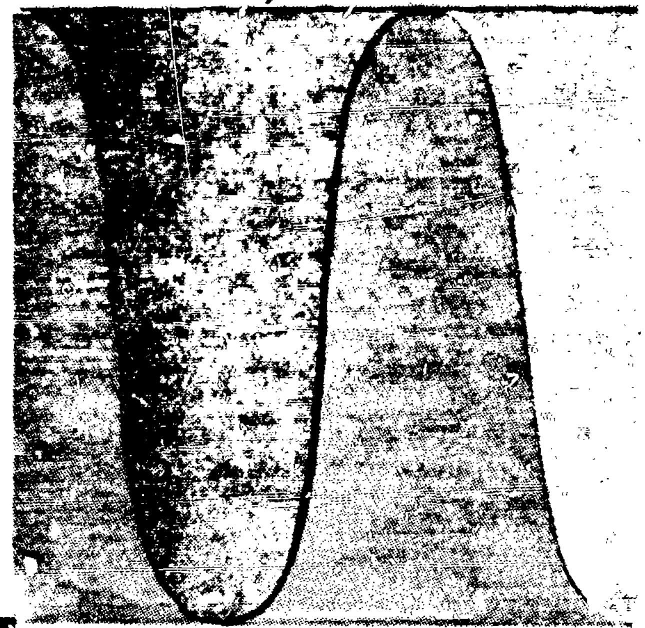

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BLAST WAVE THEORY

Akira Sakurai

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ABSTRACT

A method to find approximate solution of an idealized model of the blast wave problem is developed, the gasdynamic motion is initiated by an instantaneous energy release from a point source (or line, or plane source). Transformation of variables is introduced with a similarity parameter x as an independent variable along with a quantity less sensitive to the phenomenon. This reduces the fundamental system of equations of the problem to a manageable form for the approximation, which is conveniently performed in a power series expansion in the variable y and the coefficients of the expansion can be determined successfully from the systems of ordinary differential equations. The method has been applied to many problems of the similar type, with some modifications necessary for each case, to blast wave, the propagation of the blast wave in the non-uniform medium, exploding wire phenomenon, magnetohydrodynamics cavitation and thunder.

BLAST WAVE THEORY

Akira Sakurai

1. Introduction

A blast wave is a rather common phenomenon usually experienced as a "shock" when some explosion occurs, and the phenomenon itself is simply a kind of disturbance in the atmosphere like a sound wave. The characteristics of a blast wave are, however, quite different in many ways from those of ordinary sound waves. Unlike the velocity of sound c, the velocity of blast wave U is not constant and is always bigger than c; usually U is very large near the source of the explosion and decreases very quickly, approaching the sound velocity c as shown in Figure 1. The fact implies also the energy dissipation is more significant in a blast wave than in a sound wave. Secondly, a blast wave is not really a wave of periodic type like ordinary sound, but consists of a single pulse distinguished by the presence of the shock wave (The terminology "shock wave" is used here to indicate the front surface, not the whole pulse region which we refer to as a "blast wave"). At the front, the pressure p, the density ρ, and so forth, jump abruptly from their values at the undisturbed atmosphere $p_0, \rho_0, \ldots$ (Figure 2). Moreover, its wave form changes shape in the course of propagation in quite a different way from that in the sound wave case.

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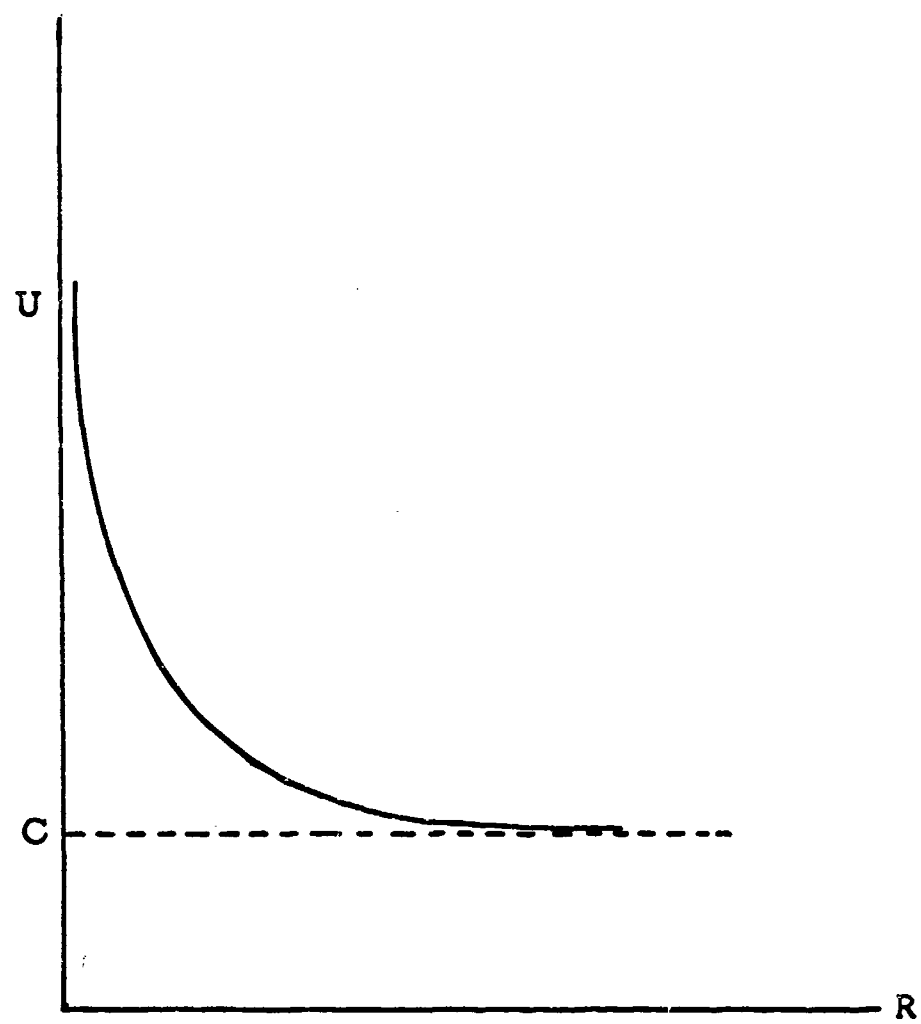

Figure 1

Decay of Blast Wave

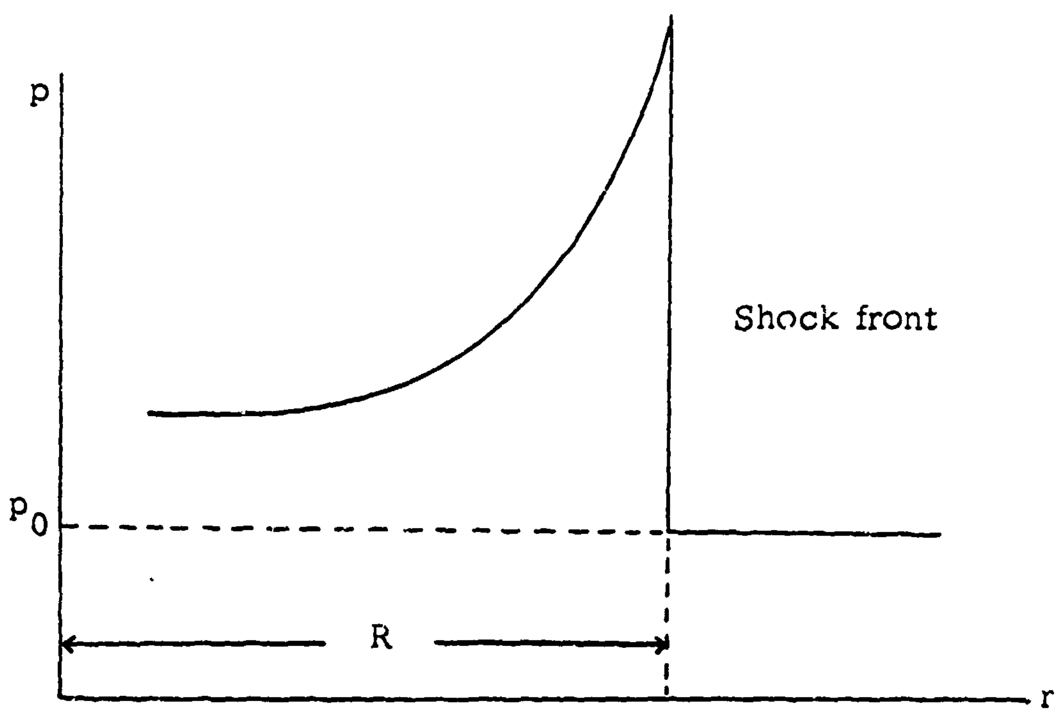

Figure 2

Pressure distribution in blast wave

These features result from the non-linearity of the phenomenon, causing the atmosphere to be so disturbed that the pressure, the density, and so forth, are considerably different from their undisturbed values p_0, ρ_0, ... In the sound wave case, on the other hand, these differences are small and the phenomenon can be treated much more simply on a linearized basis.

Although there has been a need to clarify the details of these features for many years, chiefly with the obvious aim of estimating the effects of explosives, the attempts at theoretical study of the phenomenon had to face the very difficult problem of finding the solution to the non-steady flow of the full non-linear hydrodynamic equations, satisfying a moving boundary condition at the shock front.

This situation, however, has been eased to some extent since the Second World War, the demands of which obviously stimulated renewed interest in the phenomenon. As a consequence a large amount of experimental as well as theoretical work was performed. Considerable progress has been made not only in this particular area, but also in related fields. The investigation has since been extended to quite different fields by applying the techniques and the results proved useful in the original problem.

The greatest progress was made probably when the concept of similarity was introduced, which simplified the problem while retaining the essential nature of non-linearity. The concept itself is not new, but has been familiar also in some other fields of fluid dynamics, such as boundary layer theory, conical flow theory and so forth. The assumption of similarity causes the number of independent variables to be decreased and thus frequently reduces the

fundamental partial differential equations to the more manageable ordinary differential equations.

The blast wave solution of the equations of gas dynamics was found (Sedov, 1946, Taylor, 1950) in the form of similarity solution. However the existence of a group of similarity solutions of progressive wave type had been known before in connection with the contracting spherical wave problem. (Guderley, 1942). It should be noted that the similarity in the blast wave phenomena does not hold exactly, but is valid only while the wave is strong enough to neglect the effect of the ambient atmospheric pressure. Since it is obvious that the hydrodynamical representation of the phenomenon does not hold very near the explosion source, the range where the similarity solution is valid, is accordingly limited in a very small region (even in the strong blast wave from an atomic explosion, it is reported to extend between 20 and 180m (Taylor, 1950)). Sometimes in the case of ordinary explosions there is no such region. Nevertheless, the concept of the similarity seems to be very useful since the phenomenon preserves the similarity in some extent (Sakurai, 1953, 1954) even at the stage where the front shock becomes weaker.

It has been found that the same kind of similarity solutions could be obtained for other kinds of explosions (Sedov, 1946, Sakurai, 1953, 1954, Lin, 1954) such as from a line source and a plane source, while the ordinary explosion is considered as that due to a point source. Although the blast wave from the line or plane source seems to be rather artificial in actual explosions, these solutions themselves proved to be very useful in many other different fields, especially the application of the theory to the hypersonic flow problem (Lin, 1954, Lees and Kubota,

1957) where the general principle of similitude (Hayes, 1947) between hypersonic flow and blast wave behavior had been known to hold. Most of the details on these earlier developments are seen in such books as Sedov (1957), Hayes and Probstein (1959), and the more extensive treatment on the subject given in the recent book by Korobeinikov et al. (1961).

While blast wave theory itself has been continually improved, many important applications of the theory to the various fields of research have also been taking place. It is the purpose of the present article to descirbe some of these recent developments. The fundamental idea of blast wave theory itself will be described in Chapter 2, which is essentially concerned with the transformation of the variables - both dependent and independent - based on the concept of similarity, and reduces the fundamental system of hydrodynamic equations to a more manageable form, although they are still non-linear partial differential equations.

Different methods of finding approximate solutions to the system of equations will be discussed in some detail. Various applications of the solutions thus obtained are given in the next Chapter 3. Firstly, they will be applied to the blast wave problem itself (Section 3.1); this proved to be quite successful, especially in describing the strong blast phenomenon resulting from the atomic explosion.

Suppose that we have an explosion of very large scale, as in astrophysical phenomena. It would then be necessary to modify the theory so as to include the effects of non-uniformity in initial fields caused, for example, by gravity, (Section 3.2). [Applications to the hypersonic flow problems mentioned

previously are of course indispensable, but the subject may be omitted here since this should be treated more extensively in an independent article.]

The electrical disintegration of fine wire known as EWP (Exploding Wire Phenomenon) has been attracting considerable attention in connection with various fields (see, Chace and Moore, 1959, 1962, 1964). The exploding fine wire may be regarded as an example of the line source and may produce a similar situation to that expected from the blast wave solution above (Section 3.3).

In the vast field of magnetohydrodynamics (Section 3.4), there are many problems on magnetohydrodynamic shock waves which show blast wave-like characteristics. Since extra terms are needed to describe the effects of the magnetic field, the genuine blast wave theory above is usually not applied directly to these phenomena, but needs to be modified. Some of the problems are, however, very similar to those of ordinary blast wave type and need only a slight modification, introducing the magnetic pressure. There is an interesting application of the theory to the problem involving singularity. This is given in Section 3.5 in connection with the problem of the collapse of an empty cavity in water. The procedure to this case is not straightforward because of the singularity but can be modified by introducing Lighthill's technique (1949). Quite recently application of the blast wave theory from the line source has appeared in the problem of thunder. Observations of this have revealed that the duration between the lightning and the succeeding thunder, its wave form, and so forth, are very different from the results expected from the

theory of sound. A more sophisticated way of treating the phenomenon is required to take into account its non-linearity, and to regard it as a blast wave from the lightning as a line source (Section 3. 6).

Apart from the applications mentioned, blast wave theory is mathematically a kind of transformation of variables to another set where some of the independent variables are not sensitive. The extension of the theory along the line as a technique to handle the complicated problems especially of non-linear nature, could be very useful and the development of the technique itself may be an interesting problem for the future.

§2. Blast Wave Theory

2.1 Fundamental Equations

Suppose that we have an explosion, following which there may exist for a while a very small region filled with hot matter at high pressure, which starts to expand outwards with its front headed by shock wave. The process usually takes place in a very short time after which an advancing shock wave develops, which is continuously absorbing the ambient air into the blast wave. Although some of the explosive products may still remain near the center, the amount of the air absorbed increases with time, and the later behavior of the blast wave may well be represented by the following model of the shock wave at the front and the purely gas-dynamic treatment of the motion of the air inside, which is assumed to bear ideal and non-viscous adiabatic exponent γ .

The three types of blast wave (spherical, cylindrical and plane) all have features in common and may be conveniently treated in the following unified manner. The equations of motion, continuity and energy

of the gas behind a blast wave of one of these types may be written

$$\frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial r}, \tag{2.1}$$

$$\frac{D\rho}{Dt} = -\rho\left(\frac{\partial u}{\partial r} + \frac{\alpha u}{r}\right), \tag{2.2}$$

$$\frac{D}{Dt} p\rho^{-\gamma} = 0, \tag{2.3}$$

where u is the particle velocity, p is the pressure, ρ is the density and u, p, ρ are functions of the Eulerian coordinate r (measured from the center) and the time t (measured from the instant of explosion). The coefficient α has the values:

$\alpha = 0$ for a plane wave,

$\alpha = 1$ for a cylindrical wave,

$\alpha = 2$ for a spherical wave,

and the expression D/Dt denotes

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial r} .$$

Equation (2.3) is conveniently transformed into

$$\frac{Dp}{Dt} = -\gamma p\left(\frac{\partial u}{\partial t} + \frac{\alpha u}{r}\right), \tag{2.4}$$

where we have used the equation (2.2).

The position of the shock front is represented by R (measured also from the center), which is supposed to be a monotonically increasing function of t and is related with the shock velocity U by

$$\frac{dR}{dt} = U . \tag{2.5}$$

At the shock front where $r = R$, the quantities such as u, p, ρ measure suddenly from their corresponding values in the atmosphere (presumed at rest) with pressure and density p_0, ρ_0 . The discontinuity conditions is given by the Rankine-Hugoniot relations (see for example, Courant and Friedricks, 1948) which for the present purposes, are most conventiently written

$$(u)_{r=R} = \frac{2}{\gamma+1} \frac{U^2-C^2}{U}$$

$$(p)_{r=R} = p_0 \left\{\frac{2\gamma}{\gamma+1}\left(\frac{U}{C}\right)^2 - \frac{\gamma-1}{\gamma+1}\right\}, \qquad (2.6)$$

$$(\rho)_{r=R} = \rho_0 \frac{\gamma+1}{\gamma-1}\left\{1 + \frac{2}{\gamma-1}\left(\frac{C}{U}\right)^2\right\}^{-1},$$

where C is the sound velocity given by $C = \sqrt{\gamma \frac{p_0}{\rho_0}}$. (2.7)

Now we have three equations (2.1), (2.2), (2.3) and the boundary conditions (2.6) at $r = R$. We need another condition to determine U (or R) as a function of t . The condition may be postulated in various ways depending on the feature of the explosion. We may assume, for example, a small region of high temperature and pressure at $t = 0$, from which blast wave is started. Some conditions of this type are useful especially, for the purpose of finding the solution by numerical procedure, where all the complications can be taken into account under the specified circumstances.

To describe the general features of the phenomenon, the point source model (line or plane source for $\alpha = 0, 1$) has been considered simple and appropriate for the purpose. It is assumed that a certain amount of energy is released instantaneously from a point (or line, or plane). Since the total amount of energy (kinetic and thermal) carried by the blast wave is constant and must

always be equal to the energy released, the assumption of the instantaneous release of the energy is conveniently represented in the following way (Taylor, 1950).

$$E_\alpha = \int_0^R \left(\frac{1}{2}\rho u^2 + \frac{p-p_0}{\gamma-1}\right) r^\alpha dr \ , \tag{2.8}$$

or

$$E_\alpha = \int_0^R \left(\frac{1}{2}\rho u^2 + \frac{p}{\gamma-1}\right) r^\alpha dr - \frac{p_0}{\gamma-1}\frac{R^{\alpha+1}}{\alpha+1} \ ,$$

where E_α is related to the released energy as

$$E_\alpha = \begin{cases} \text{Explosion energy per unit area} \quad \text{for } \alpha = 0 \ , \\ (\text{Explosion energy per unit line})\ (2\pi)^{-1} \quad \text{for } \alpha = 1 \ . \\ (\text{Explosion energy})\ (4\pi)^{-1} \quad \text{for } \alpha = 2 \ . \end{cases} \tag{2.9}$$

Although the model is apparently inadequate to represent the actual situation in the very early stages, where all kinds of complications are involved, it should be recalled that in this period, the gasdynamical representation of the phenomenon is not adequate in any case. The advantage of using the model is two-fold. Firstly, the model, although it is not adequate in the earlier stage, becomes more and more accurate at the later stages <u>regardless of the kind of the explosive or the feature of the explosion.</u> Secondly, the shock front appears from the beginning in this model, since the explosion is assumed to take place instantaneously at a point (or line or plane). Thus we may avoid the nuisance of the shock formation and at the same time we may expect a kind of similarity in the flow field. Because of this similarity, the whole system of equations can be transformed to a mathematically feasible system and thus can be solved approximately, which will be seen in the following Section 2.2.

It should be noticed that since an instantaneous source is assumed, the second shock may not be developed in this model. While the second shock or more shocks appear in actual explosions and they themselves provide interesting effects, their features depend considerably on the sequence of the explosion and are well understood in connection with specified initial conditions. It seems more appropriate to use direct method (for example, Holt, 1956) or numerical method (for example, Goldstein and v. Neumann, 1955, Brode, 1959) for studying their features in individual cases.

2.2. Blast Wave Transformation

Let us introduce new independent variables (x, y) defined by

$$\frac{r}{R} = x, \quad \frac{c^2}{U^2} = y \ , \qquad (2.10)$$

and transform the dependent variables as

$$u = Uf(x, y), \quad p = p_0 y^{-1} g(x, y), \quad \rho = \rho_0 h(x, y), \qquad (2.11)$$

where f, g, h are new dependent variables to be found (Sakurai, 1953).

The introduction of x is most important and is based on the similarity nature of the one-dimensional flow given by Eqs. (2.1), (2.2), and (2.3), for which a group of similarity solutions depending only on x exists (See for example, Sedov, 1957). Although none of these solutions satisfies the conditions(2.6) and (2.8) exactly, there is a ...on satisfying them approximately when the front shock is strong and $(p_0/p)_{r=R}$ is negligible. Since the front shock decays very quickly even in the stronger explosions, i' is necessary to take into account the effect of the pressure neglected above, for which we need to retain independent variable other than x. The variable y in (2.10) is

chosen so as to fit the conditions (2.6) and (2.8), which are now written as:

$$\begin{cases} f(1,y) = \frac{2}{\gamma+1}(1-y) , \\ g(1,y) = \frac{2\gamma}{\gamma+1} - \frac{\gamma-1}{\gamma+1} y , \\ h(1,y) = \frac{\gamma+1}{\gamma-1}(1 + \frac{2}{\gamma-1} y)^{-1} , \end{cases} \tag{2.12}$$

and

$$y(\frac{R_0}{R})^{\alpha+1} = \int_0^1 (\frac{\gamma}{2} hf^2 + \frac{g}{\gamma-1}) x^{\alpha} dx - \frac{\gamma}{(\alpha+1)(\gamma-1)} , \tag{2.13}$$

where we have put

$$R_0 = (\frac{E_\alpha}{P_0})^{1/(\alpha+1)} \tag{2.14}$$

The transformations (2.11) are thus partly chosen to fit the conditions (2.6) and (2.8), and at the same time, to match the requirements of the similarity solution.

Since $y \propto (p_0/p)_{r=R}$ for a strong shock (See Eq. (2.6)), the solution tends to the similarity solution as $y \to 0$ ($R \to 0$). Also as R tends to ∞, y tends to 1 ($U \to C$). The condition (2.12) is not much affected by the value of y, which varies only between 0 and 1 and we may expect a similar insensitivity to y in condition (2.13) also. Thus the variable y is expected to have little effect in the solution. As a matter of fact, we can make other choices of the variable such as R (See Section 3.2 below), which are expected to be insensitive to the solution.

By introducing (2.10) and (2.11), the fundamental equations (2.1), (2.2), (2.3) are now transformed into

$$
\begin{cases}
-\frac{1}{2}\lambda f + (f - x)\frac{\partial f}{\partial x} + \lambda y \frac{\partial f}{\partial y} = \frac{-1}{\gamma h}\frac{\partial g}{\partial x} , \\
(f - x)\frac{\partial h}{\partial x} + \lambda y \frac{\partial h}{\partial y} = -h\left(\frac{\partial f}{\partial x} + \frac{\alpha f}{x}\right) , \\
-\lambda g + (f - x)\frac{\partial g}{\partial x} + \lambda y \frac{\partial g}{\partial y} = -\gamma g\left(\frac{\partial f}{\partial x} + \frac{\alpha f}{x}\right) ,
\end{cases}
\tag{2.15}
$$

where

$$
\lambda = \frac{R}{y}\frac{dy}{dR} . \tag{2.16}
$$

and λ is considered as a function of y only, in fact R is represented by Eq. (2.13) as a function of y and we find, by differentiation of Eq. (2.13) with respect to y ,

$$
\lambda = \left[(\alpha+1)J - \frac{1}{\gamma-1}y\right]\left(J - y\frac{dJ}{dy}\right)^{-1} , \tag{2.17}
$$

where we have put

$$
\int_0^1 \left(\frac{\gamma}{2}hf^2 + \frac{g}{\gamma-1}\right)x^{\alpha}\,dx = J . \tag{2.18}
$$

Although the system (2.15) includes the integral J in (2.18) through λ in (2.17) and still looks formidable, it will be seen in the next section that the system is much more easier to treat than the original one because of its insensitiveness to y .

It is also worthwhile to note that the system is represented by three non-dimensional numerical constants α, γ and R_0, in which R_0 is most important and is related to the scale of the explosion.

2.3. Solution in a Series Expansion in y

2.3.1 General procedure

It is readily seen that if we put $y = 0$ in Eq. (2.15) it is reduced to a system of ordinary differential equations where λ is constant and equal to $\alpha+1$,

while the condition (2.12) for this case provides the boundary values at $x = 1$ and Eq. (2.13) gives the relation between y and R (or U and R) as $y \propto R^{\alpha+1}$. These are all the immediate results of the <u>similarity solution of the intense explosion</u>. Now it will be shown for general y that if we express f, g, h, in power series of y as,

$$\begin{cases} f = f^{(0)} + y f^{(1)} + y^2 f^{(2)} + \dots , \\ g = g^{(0)} + y g^{(1)} + y^2 g^{(2)} + \dots , \\ h = h^{(0)} + y h^{(1)} + y^2 h^{(2)} + \dots , \end{cases} \tag{2.19}$$

where $f^{(i)}$, $g^{(i)}$, $h^{(i)}$ $(i = 0, 1, 2, \dots)$ are all assumed as functions of x only, then these $f^{(i)}$, $g^{(i)}$, $h^{(i)}$ $(i = 1, 2, \dots)$ can be determined successively starting from the similarity solution given by $f^{(0)}$, $g^{(0)}$, $h^{(0)}$ (Sakurai, 1953, 1954). It should be noticed that this is not always possible in a non-linear system, but often quantities for $i = 1$ or more appear in the first approximation, which makes the successive approximation procedure impossible. Using the expression (2.19), the integral J defined in Eq. (2.18) becomes formally,

$$J = J_0 (1 + \sigma_1 y + \sigma_2 y^2 + \dots) , \tag{2.20}$$

where we have put

$$\begin{cases} J_0 = \int_0^1 \left(\frac{\gamma}{2} h^{(0)} f^{(0)2} + \frac{g^{(0)}}{\gamma-1}\right) x^\alpha dx , \\ \sigma_1 J_0 = \int_0^1 \left(\gamma h^{(0)} f^{(0)} f^{(1)} + \frac{\gamma}{2} f^{(0)2} h^{(1)} + \frac{g^{(1)}}{\gamma-1}\right) x^\alpha dx , \\ \sigma_2 J_0 = \int_0^1 \left(\gamma h^{(0)} f^{(0)} f^{(2)} + \frac{\gamma}{2} f^{(0)2} h^{(2)} + \frac{g^{(2)}}{\gamma-1}\right) x^\alpha dx \\ \qquad \dots \qquad + \frac{\gamma}{2} \int_0^1 \left(h^{(0)} f^{(1)2} + 2h^{(1)} f^{(1)} f^{(0)}\right) x^\alpha dx , \end{cases} \tag{2.21}$$

With use of expression (2.20), Eq. (2.13) becomes

$$y\left(\frac{R_0}{R}\right)^{\alpha+1} = J_0\left[1 + \left\{\sigma_1 - \frac{1}{J_0(\alpha+1)(\gamma-1)}\right\} y + \sigma_2 y^2 + \ldots\right], \tag{2.22}$$

which shows the y-R relation more clearly. λ in Eq. (2.17) is also expanded as

$$\lambda = (\alpha + 1)(1 + \lambda_1 y + \lambda_2 y^2 + \ldots) \,, \tag{2.23}$$

where we have put

$$\begin{cases} \lambda_1 = \sigma_1 - \dfrac{1}{J_0(\alpha+1)(\gamma-1)} \\ \lambda_2 = 2\sigma_2 \,, \\ \ldots \end{cases} \tag{2.24}$$

Putting Eqs. (2.19) and (2.23) into Eq. (2.15) and equating the coefficients of the terms in corresponding powers of y, we get the following systems of differential equations.

From the coefficients of the y° terms,

$$\begin{cases} (f^{(0)} - x)\, h^{(0)} f^{(0)\prime} + \dfrac{1}{\gamma} g^{(0)\prime} = \dfrac{\alpha+1}{2} f^{(0)} h^{(0)} \,, \\ f^{(0)\prime} + \dfrac{\alpha}{x} f^{(0)} = (x - f^{(0)}) \dfrac{h^{(0)\prime}}{h^{(0)}} \,, \\ \gamma\left(f^{(0)\prime} + \dfrac{\alpha}{x} f^{(0)}\right) - \alpha - 1 = (x - f^{(0)}) \dfrac{g^{(0)\prime}}{g^{(0)}} \,, \end{cases} \tag{2.25}$$

From the coefficients of the y terms,

$$
\begin{cases}
h^{(0)}(f^{(0)}-x)f^{(1)\prime} + \frac{1}{\gamma} g^{(1)\prime} = -(\frac{\alpha+1}{2} + f^{(0)\prime})h^{(0)} f^{(1)} \\
\qquad + \{\frac{\alpha+1}{2} f^{(0)} + (x-f^{(0)})f^{(0)\prime}\}h^{(1)} + \frac{\alpha+1}{2}\lambda_1 f^{(0)} h^{(0)} , \\
h^{(0)} f^{(1)\prime} - (x-f^{(0)})h^{(1)\prime} = -(h^{(0)\prime} + \frac{\alpha}{x} h^{(0)})f^{(1)} - (f^{(0)\prime} + \frac{\alpha}{x} f^{(0)} + \alpha+1)h^{(1)} , \\
\gamma g^{(0)} f^{(1)\prime} - (x-f^{(0)})g^{(1)\prime} = -(g^{(0)\prime} + \frac{\alpha\gamma}{x} g^{(0)})f^{(1)} - \gamma(f^{(0)\prime} + \frac{\alpha}{2} f^{(0)}) \\
\qquad + (\alpha+1)\lambda_1 g^{(0)} , \\
\cdots .
\end{cases}
\tag{2.26}
$$

Similarly Eq. (2.12) yield the following conditions:

$$
\begin{cases}
f^{(0)}(1) = \frac{2}{\gamma+1}, \quad g^{(0)}(1) = \frac{2\gamma}{\gamma+1}, \quad h^{(0)}(1) = \frac{\gamma+1}{\gamma-1} ; & (2.27) \\
f^{(1)}(1) = -\frac{2}{\gamma+1}, \quad g^{(1)}(1) = -\frac{\gamma-1}{\gamma+1}, \quad h^{(1)}(1) = -\frac{2(\gamma+1)}{(\gamma-1)^2} ; & (2.28) \\
f^{(2)}(1) = 0, \quad g^{(2)}(1) = 0, \quad h^{(2)}(1) = \frac{4(\gamma+1)}{(\gamma-1)^3} ; & (2.29) \\
\cdots .
\end{cases}
$$

The first system of non-linear differential equations (2.25) with the conditions (2.27) gives the similarity solution (given by Taylor (1950) and Sedov (1946)), and the value of the integral J_0 in Eq. (2.21) is determined from these $f^{(0)}$, $g^{(0)}$, $h^{(0)}$. The second system of Equations (2.26), after inserting the values of $f^{(0)}$, $g^{(0)}$, $h^{(0)}$ obtained above, becomes a system of linear inhomogeneous differential equations for $f^{(1)}$, $g^{(1)}$, $h^{(1)}$. The system also includes the unknown parameter λ_1, which is related to σ_1, by the relation (2.24). The solutions $f^{(1)}$, $g^{(1)}$, $h^{(1)}$ of Eq. (2.26) subject to the

boundary condition (2.28) thus include λ_1 (or σ_1). Inserting these solutions in the right side of the second equation of Eqs. (2.21), we have an equation to determine λ_1. With this value of λ_1, $f^{(1)}$, $g^{(1)}$, $h^{(1)}$ are determined finally, and we have the second approximation to the problem to the order of y, in the following form,

$$\left\{\begin{array}{l} u = U\,\{f^{(0)} + y\,f^{(1)} + O(y^2)\}\ , \\ p = (p_0/y)\ \{g^{(0)} + y\,g^{(1)} + O(y^2)\}\ , \\ \rho = \rho_0\ \{h^{(0)} + y\,h^{(1)} + O(y^2)\}\ , \\ (\frac{C}{U})^2\,(\frac{R_0}{R})^{\alpha+1} = J_0\,[1 + \lambda_1 y + O(y^2)]\ . \end{array}\right. \tag{2.30}$$

The procedure in the third and further approximations is the same in principle as in the second approximation above, and $f^{(i)}$, $g^{(i)}$, $h^{(i)}$, λ_i for all i are to be found successively.

While it is naturally not easy to see the validity of the solution in expanded form, some verifications will be obtained simply by comparison with corresponding experimental data as well as with some purely numerical solutions. It is noted that singularities of the form y^{-1} in p or $y^{\lambda(\alpha+1)}$ in R (see Eq. (2.30)) are removed before the expansion, where only the parts presumably insensitive to y are concerned. It is not likely, however, that the expansion is good as well up to $y = 1$, where all quantities are subject to another kind of singularity (see Section 2.4 below).

In fact it is expected that $R \to \infty$ as $y \to 1$ and accordingly from Eqs. (2.13) and (2.18), J must approach the value $(\alpha+1)^{-1}(\gamma-1)^{-1}$ as $y \to 1$, and it is hardly to be expected that the expansions of J in Y behaves in this way. The

difficulty will be overcome to some extent in the following Section 2.4. Another technical way of improving the expansion procedure is to expand ρ_0/ρ instead of ρ/ρ_0 as in Eq. (2.19), which led to the expansion of $[1+2y/(\gamma-1)]^{-1}$ in $2y/(\gamma-1)$ (see Eq. (2.12)). Since $2/(\gamma-1)$ is usually rather a large number for ordinary values of γ near for $\gamma=1.4$, the expansion in $2y/(\gamma-1)$ is expected to be poor and the alternative way of expanding ρ_0/ρ is supposed to give better results.

2.3.2 First approximation

Equations (2.25) are known to have two intermediate integrals and the resulting first order differential equation is integrated exactly. Multiplying the second equation of Eqs. (2.25) by $(\gamma-1)$ and subtracting the third one yields,

$$(x-f^{(0)})\{(\gamma-1)\frac{h^{(0)'}}{h^{(0)}}-\frac{g^{(0)'}}{g^{(0)}}\} = 1-f^{(0)'}+\frac{\alpha}{x}(x-f^{(0)}) ,$$

which is readily integrated to give

$$g^{(0)}(x-f^{(0)})h^{(0)^{-(\gamma-1)}}x^{\alpha} = \frac{2\gamma}{\gamma+1}(\frac{\gamma-1}{\gamma+1})^{\gamma} , \qquad (2.31)$$

where we have used the condition (2.27) to determine the integration constant. The second integral is, from the energy relation,

$$\frac{\partial}{\partial t}[r^{\alpha}(\frac{p}{\gamma-1}+\frac{\rho u^2}{2})]+\frac{\partial}{\partial r}[r^{\alpha}u(\frac{\gamma p}{\gamma-1}+\frac{\rho u^2}{2})] = 0 ,$$

which, in the present case, is reduced to

$$\frac{d}{dx}[-x^{\alpha+1}\{\frac{g^{(0)}}{\gamma(\gamma-1)}+\frac{1}{2}f^{(0)^2}h^{(0)}\}+x^{\alpha}f^{(0)}(\frac{g^{(0)}}{\gamma-1}+\frac{1}{2}f^{(0)^2}h^{(0)})] = 0 ,$$

and integrated to give

$$f^{(0)} = x(\frac{f^{(0)^2}}{2}+\frac{1}{\gamma}\frac{1}{\gamma-1}\frac{g^{(0)}}{h^{(0)}})(\frac{f^{(0)^2}}{2}+\frac{1}{\gamma-1}\frac{g^{(0)}}{h^{(0)}})^{-1} , \qquad (2.32)$$

where we have again used the condition (2.27). Eliminating $g^{(0)}$ and $h^{(0)}$ from Eqs. (2.31), (2.32) and the third equation of Eq. (2.25), we get

$$\frac{x-f^{(0)}}{2-\gamma}\left\{\frac{\alpha}{x}+\frac{2f^{(0)\prime}}{f^{(0)}}+2\frac{1-f^{(0)\prime}}{x-f^{(0)}}-\frac{\gamma f^{(0)\prime}-1}{\gamma f^{(0)}-x}\right\}=f^{(0)\prime}+\frac{\alpha}{x}f^{(0)} \quad ,$$

which is evidently integrated by putting $f^{(0)}/x = F$(say) with the result

$$2\log f^{(0)} = -\frac{\alpha(\alpha+3)(\gamma-1)}{(\alpha+1)\gamma-(\alpha-1)}\log x+\frac{(\alpha+1)(\gamma-1)}{2\gamma+\alpha-1}\log(\gamma f^{(0)}-x)$$

$$-\frac{(\alpha^2+2\alpha+5)\gamma^2+(-3\alpha^2+2\alpha+1)\gamma+4(\alpha^2-1)}{(2\gamma+\alpha-1)\{(\alpha+1)\gamma-(\alpha-1)\}}\log\left|x-\frac{(\alpha+1)\gamma-(\alpha-1)}{\alpha+3}f\right|$$

$$+\log c \quad , \tag{2.33}$$

where c is the integration constant determined by Eq. (2.27), given as

$$\log c = 2\log\frac{2}{\gamma+1}-\frac{(\alpha+3)(\gamma-1)}{2\gamma+\alpha-1}\log\frac{\gamma-1}{\gamma+1}+\frac{(\alpha^2+2\alpha+5)\gamma^2+(-3\alpha^2+2\alpha+1)\gamma+4(\alpha^2-1)}{(2\gamma+\alpha-1)\{(\alpha+1)\gamma-(\alpha-1)\}}$$

$$\cdot\log\left|\frac{3\alpha+1-(\alpha-1)\gamma}{(\alpha+3)(\gamma-1)}\right| \quad .$$

It is also known that this kind of exact solution exists for more a general type of similarity solutions such as the initial distribution obeying a power law in distance, time dependent energy supply, and so on (for more details for example, see Korobeinikov et al., 1961). It is noticed that the solution (2.33) is not always valid but needs to be modified for some values of γ and α. For example $\gamma = 2$ is singular and the solution (2.33) is modified by applying a limiting process as $\gamma \to 2$ namely,

$$f^{(0)} = xF, \quad F^2\frac{(1-F)^2}{2F-1} = \frac{4}{27}x^{-\alpha-3} \quad . \tag{2.34}$$

$\alpha = 2$, $\gamma = 7$ is also singular and the solution is simply given by

$$f^{(0)} = \frac{x}{4}, \quad g^{(0)} = \frac{7}{4}x^3, \quad h^{(0)} = \frac{4}{3}x \ . \tag{2.35}$$

The case $\gamma = 2$ is interesting because plasma under uniform magnetic field is expected to behave somewhat like a gas with $\gamma = 2$ (Spitzer, 1956), while $\gamma = 7$ is an appropriate value for water. Graphs of $f^{(0)}(x)$, $g^{(0)}(x)$, $h^{(0)}(x)$ in the typical case of $\gamma = 1.4$ (air) are shown in Figure 3. Little divergence

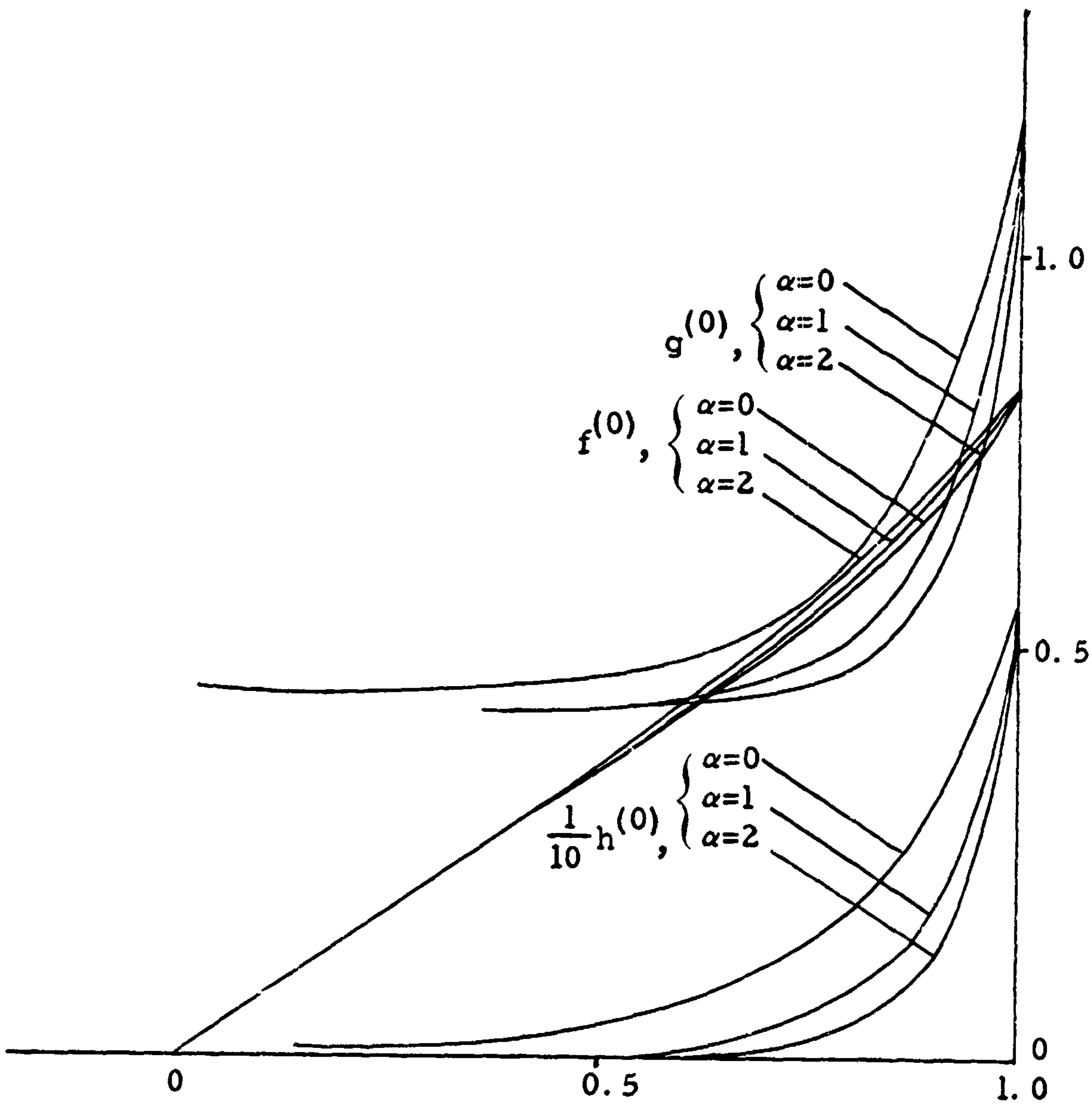

Figure 3

Solution curves of $f^{(0)}(x)$, $g^{(0)}(x)$, $h^{(0)}(x)$ for $\alpha = 0, 1, 2$ taking $\gamma = 1.4$

is found for various values of γ . $f^{(0)}$ at $x = 0$ is always zero, showing

$$u(0,t) = 0 \tag{2.36}$$

as required, a condition sometimes used to determine the flow field for further approximations (see below in Section 2.3.4).

After obtaining the $f^{(0)}$, $g^{(0)}$, $h^{(0)}$ values above, the $\gamma - R$ relation (or U-R relation) given in Eq. (2.22) is obtained in the first approximation by evaluating the integral J_0 in Eq. (2.21). Unfortunately, J_0 cannot be integrated analytically, but is only found numerically. Since the exact expression for $f^{(0)}$, $g^{(0)}$, $h^{(0)}$ given by Eqs. (2.31), (2.32), (2.33) are rather inconvenient to compute it is sometimes more efficient to get their values by direct numerical integration of the original system (2.25) starting from the values given in Eq. (2.27) at $x = 1$. The most recent numerical data on $f^{(0)}$, $g^{(0)}$, $h^{(0)}$ are given by Jones (1962) for some values of γ . J_0 values computed from these data are exact to three digits when compared with the values obtained by various methods. Some of the values are reproduced in Table I. The divergence between these values indicates the difficulty in finding the numerical values of $f^{(0)}$, $g^{(0)}$, $h^{(0)}$ partly because of their rapid change near $x = 1$.

2.3.3 Further approximations

It is convenient to introduce new variables $\varphi^{(i)}$, $\psi^{(i)}$, $\chi^{(i)}$ $(i = 1, 2, \ldots)$ by the relations,

$$f^{(i)} = (x - f^{(0)})\, \varphi^{(i)}, \quad g^{(i)} = g^{(0)} \psi^{(i)}, \quad h^{(i)} = h^{(0)} \chi^{(i)} \quad . \tag{2.37}$$

Equations (2.26) to the second approximation are thus reduced to

Table I. J_0 Values

α	2 (Spherical)				1 (Cylindrical)			0 (Plane)		
γ	5/3	7/5	1.3	6/5	5/3	7/5	6/5	5/3	7/5	6/5
Taylor (1950)	0.404	0.596	0.755	1.031						
Sakurai (1953)					0.585	0.877	1.547	1.137	1.696	3.024
Lin (1954)						0.858				
Sedov (1957)	0.412	0.593	0.718	1.041	0.584	0.891	1.558	1.125	1.708	2.940
Rogers (1958)		0.597		1.031		0.898	1.547		1.708	3.072
Rouse (1959)						0.883				
Gerber & Bartos (1961)					0.600	0.878				
Jones (1962)	0.408	0.594			0.600	0.878		1.130	1.708	

$$-(x - f^{(1)})\varphi^{(1)'} + \frac{g^{(0)}}{\gamma h^{(0)}(x - f^{(0)})}\psi^{(1)'} = -(2f^{(0)'} + \frac{\alpha-1}{2})\varphi^{(1)} \tag{2.38}$$

$$+ (f^{(0)'} + \frac{\alpha+1}{2}\frac{f^{(0)}}{x-f^{(0)}})(\chi^{(1)} - \psi^{(1)}) + \frac{\alpha+1}{2}\frac{f^{(0)}}{x-f^{(0)}}\lambda_1 ,$$

$$(x - f^{(0)})(-\varphi^{(1)'} + \chi^{(1)'}) = (\alpha+1)(\varphi^{(1)} + \chi^{(1)}) \quad , \tag{2.39}$$

$$(x - f^{(0)})(-\gamma\varphi^{(1)'} + \psi^{(1)'}) = (\alpha+1)\{(\gamma-1)\varphi^{(1)} + \psi^{(1)} - \lambda_1\} \quad , \tag{2.40}$$

where we have used Eq. (2.25).

The condition (2.28) at $x = 1$ becomes

$$\varphi^{(1)}(1) = -\frac{2}{\gamma-1}, \quad \psi^{(1)}(1) = -\frac{\gamma-1}{2\gamma}, \quad \chi^{(1)}(1) = -\frac{2}{\gamma-1} \quad . \tag{2.41}$$

Another condition (2.21), eliminating σ_1 by Eq. (2.24), is transformed into

$$\int_0^1 \{\gamma f^{(0)}(x-f^{(0)})h^{(0)}\varphi^{(1)} + \frac{g^{(0)}}{\gamma-1}\psi^{(1)} + \frac{\gamma}{2}f^{(0)2}h^{(0)}\chi^{(1)}\}x^{\alpha}dx \tag{2.42}$$

$$= \lambda_1 J_0 + \frac{1}{(\alpha+1)(\gamma-1)} \quad .$$

It is readily seen that the combination of Eqs. (2.39), (2.40) written symbolically as $(2\gamma-1)\times$ Eq. (2.39) $- 2\times$ Eq. (2.40) gives

$$(x-f^{(0)})\frac{d}{dx}[\varphi^{(1)} - 2\psi^{(1)} + (2\gamma-1)\chi^{(1)}] = (\alpha+1)[\varphi^{(1)} - 2\psi^{(1)} + (2\gamma-1)\chi^{(1)} + 2\lambda_1] \quad ,$$

which is integrated to give

$$\varphi^{(1)} - 2\psi^{(1)} + (2\gamma-1)\chi^{(1)} + 2\lambda_1 = (2\lambda_1 - \frac{3\gamma-1}{\gamma}\frac{\gamma+1}{\gamma-1}) \tag{2.43}$$

$$\cdot \exp\left(\int_1^x \frac{\alpha+1}{x-f^{(0)}}dx\right) \quad ,$$

where the integration constant is determined by Eq. (2.41).

$$\int_0^1 (Q_1 \varphi^{(1)} + Q_2 \psi^{(1)}) x^{\alpha} dx + I_{01} + \lambda_1 I_{02} = \lambda_1 J_0 + \frac{1}{(\gamma-1)(\alpha+1)} , \qquad (2.45)$$

where Q_1, Q_2, I_{01}, I_{02} are given by

$$Q_1 = \gamma f^{(0)} h^{(0)} \left(x - \frac{1}{2}\frac{4\gamma-1}{2\gamma-1} f^{(0)}\right) ,$$

$$Q_2 = \frac{\gamma}{2\gamma-1} f^{(0)^2} h^{(0)} + \frac{1}{\gamma-1} g^{(0)} ,$$

$$I_{01} + I_{02} \lambda_1 = \frac{1}{2}\frac{\gamma}{2\gamma-1} \int_0^1 f^{(0)^2} h^{(0)} \left\{\left(2\lambda_1 - \frac{3\gamma-1}{\gamma}\frac{\gamma+1}{\gamma-1}\right) R - 2\lambda_1\right\} x^{\alpha} dx .$$

Although Eq. (2.44) are linear in $\varphi^{(1)}$, $\psi^{(1)}$, and λ_1, it is apparently not possible to find the solution analytically, except for the very special cases as $\alpha = 2$, $\gamma = 7$, to which the first approximation becomes very simple (see Eq. (2.35)), and P_1, P_2, P_5 are reduced to simple forms proportional only to x^{-1}; thus $\varphi^{(1)}$, $\psi^{(1)}$ are found in the form of a certain power of x. (Morawetz, 1954, see also Korobeinikov et. al., 1961). Generally, Eq. (2.44) may be integrated numerically. The equations, however contain an undetermined parameter λ_1, and are not suitable for numerical integration in their original form in Eq. (2.44), but may be integrated in the following way. First we split $\varphi^{(1)}$, $\psi^{(1)}$ into two parts as

$$\varphi^{(1)} = \varphi_1^{(1)} + \lambda_1 \varphi_2^{(1)} , \quad \psi^{(1)} = \psi_1^{(1)} + \lambda_1 \psi_2^{(1)} , \qquad (2.46)$$

where $\varphi_1^{(1)}$, $\varphi_2^{(1)}$, $\psi_1^{(1)}$, $\psi_2^{(1)}$ are independent of λ_1.

Substituting Eq. (2.46) from Eqs. (2.44) and (2.45), we get the following systems of equations,

$$\begin{cases} \psi_1^{(1)'} = P_1 \varphi_1^{(1)} + P_2 \psi_1^{(1)} + P_3 , \\ \varphi_1^{(1)'} = \frac{1}{\gamma} \psi_1^{(1)'} + P_5 \{(\gamma-1)\varphi_1^{(1)} + \psi_1^{(1)}\} , \end{cases} \qquad (2.47)$$

$$\begin{cases} \psi_2^{(1)\prime} = P_1 \varphi_2^{(1)} + P_2 \psi_2^{(1)} + P_4 \ , \\ \varphi_2^{(1)\prime} = \frac{1}{\gamma} \psi_2^{(1)\prime} + P_5 \{ (\gamma-1) \varphi_2^{(1)} + \psi_2^{(1)} - 1 \} \ , \end{cases} \tag{2.48}$$

$$\lambda_1 \{ \int_0^1 (Q_1 \varphi_2^{(1)} + Q_2 \psi_2^{(1)}) x^\alpha \, dx + I_{02} - J_0 \} \tag{2.49}$$

$$= - \int_0^1 (Q_1 \varphi_1^{(1)} + Q_2 \psi_1^{(1)}) x^\alpha \, dx - I_{01} + \frac{1}{(\gamma-1)(\alpha+1)} \ .$$

Eqs. (2.47), (2.48) may be integrated numerically from the initial values at $x = 1$, which are obtained from Eqs. (2.41) and (2.46) as,

$$\varphi_1^{(1)}(1) = -\frac{2}{\gamma-1}, \ \psi_1^{(1)}(1) = -\frac{\gamma-1}{2\gamma}; \ \varphi_2^{(1)}(1) = \psi_2^{(1)}(1) = 0 \ .$$

Using the values of $\varphi_1^{(1)}$, $\varphi_2^{(1)}$, $\psi_1^{(1)}$, $\psi_2^{(1)}$ thus obtained, λ_1 may be determined, from Eq. (2.49). Once λ_1 is determined, $\varphi^{(1)}$, $\psi^{(1)}$ can be evaluated from Eq. (2.46) and $\chi^{(1)}$ is given by Eq. (2.43). The numerical procedure has been performed for $\gamma = 1.4$ in each case of $\alpha = 0, 1, 2,$ and the values of λ_1 obtained are -2.138, -1.989, -1.918, respectively, showing these

Table II, $-\lambda_1$ Values

from Korobeinikov & Chushkin

α/γ	1.1	1.2	1.3	1.4	5/3	2	3
0	2.3257	2.2437	2.1862	2.1433	2.0683	2.0143	1.9407
1	2.0866	2.0424	2.0092	1.9836	1.9374	1.9043	1.8632
2	2.0010	1.9666	1.9396	1.9182	1.8785	1.8496	1.8141

values are almost 2. Recently the fact has been verified also for different values of γ by extensive numerical works by Korobeinikov & Chushkin (1963). These values obtained by them are listed in Table II.

Functions $\varphi^{(1)}, \psi^{(1)}, \chi^{(1)}$ for $\gamma = 1.4$ are shown graphically in Figure 4.

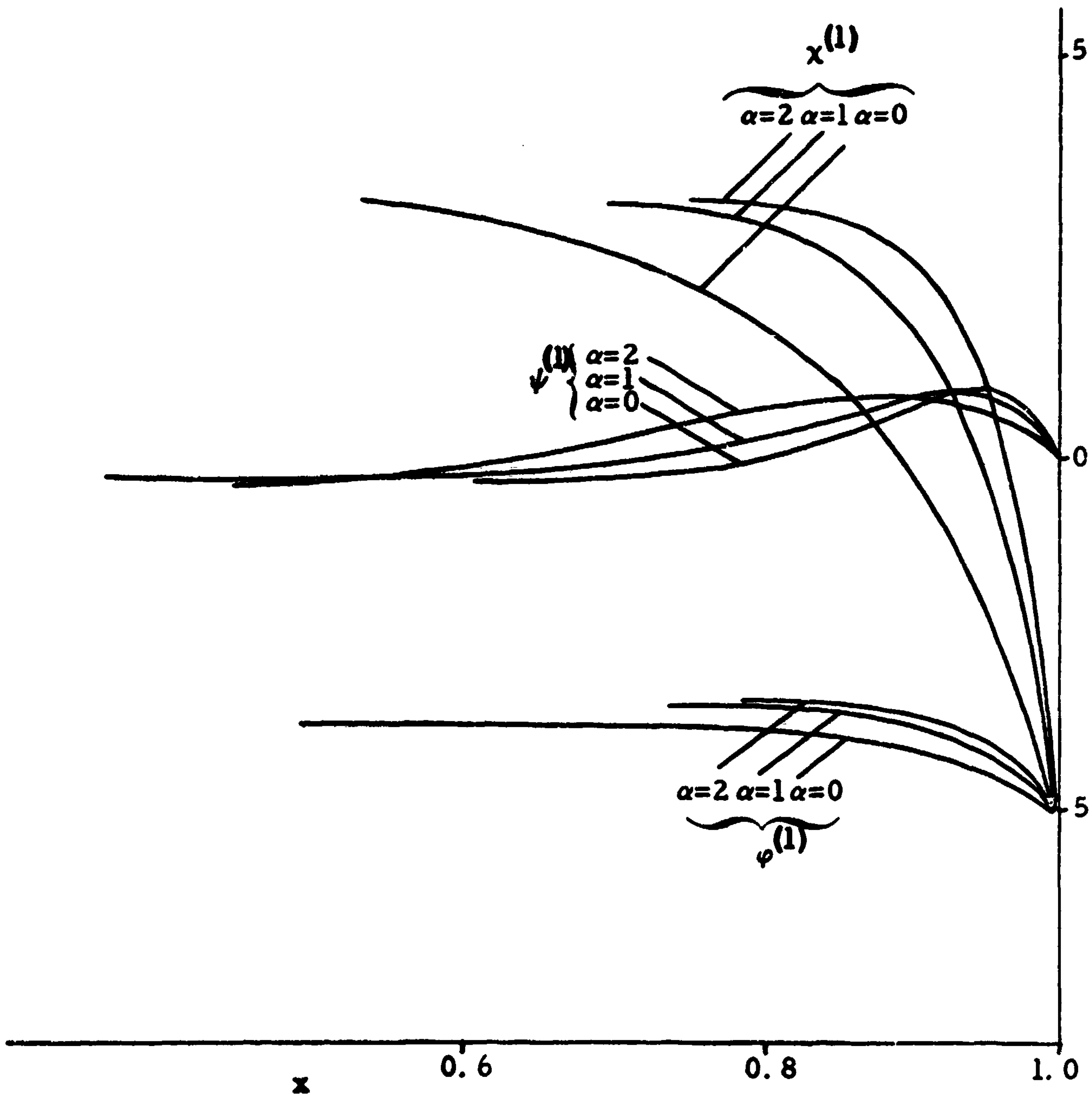

Figure 4

Solution curves of $\varphi^{(1)}, \psi^{(1)}, \chi^{(1)}$ for
$\alpha = 0, 1, 2$ taking $\gamma = 1.4$

Exactly the same procedure as above is applicable to the further stages of approximations. Swigart (1960) carried out the third approximation in the case $\gamma = 1.4$, $\alpha = 1$. The value of λ_2 obtained is

$$\lambda_2 = 2.7373 \quad (\text{for } \gamma = 1.4, \ \alpha = 1) ,$$

while the functions $\varphi^{(2)}$, $\psi^{(2)}$, $\chi^{(2)}$ are shown in Figure 5.

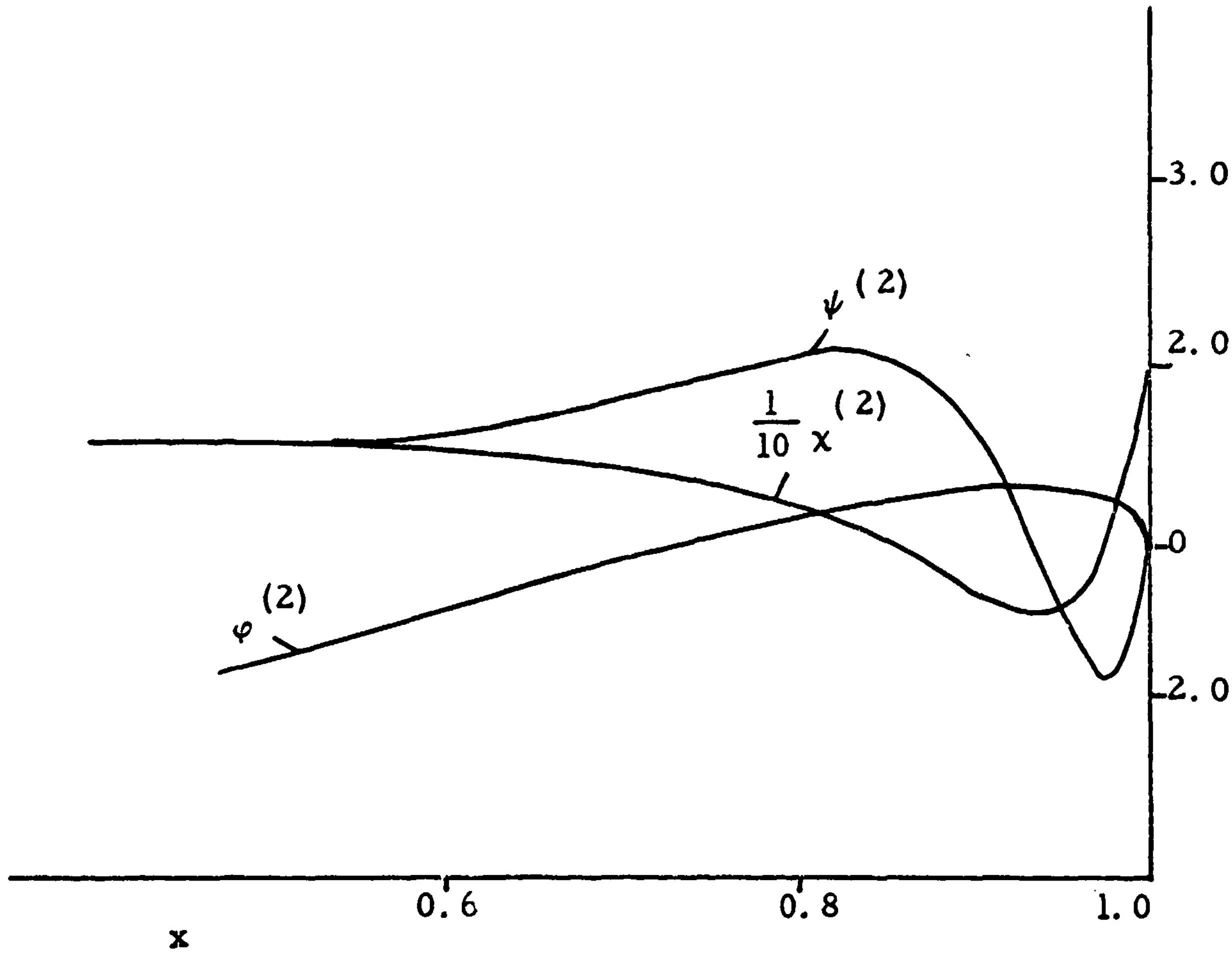

Figure 5

Solution curves of $\varphi^{(2)}$, $\psi^{(2)}$, $\chi^{(2)}$
for $\alpha = 1$ taking $\gamma = 1.4$. (Swigart, 1960)

2.3.4 Other Methods

It is seen in Figure 5 that $\chi^{(2)}$ varies considerably and its value is rather large compared with $\varphi^{(2)}$, $\psi^{(2)}$. This indicates, as suggested in the proceeding section 2.3.1., that better results will be obtained by expanding in ρ_0/ρ instead of ρ/ρ_0 . In the expansion of ρ_0/ρ, the quantity corresponding

to $\chi^{(2)}$ is zero at $x = 1$ and may behave better in the whole range.

The quantities $\varphi^{(1)}$, $\psi^{(1)}$ as well as $\varphi^{(2)}$, $\psi^{(2)}$, themselves vary slowly, but $\varphi_1^{(1)}$, $\varphi_2^{(1)}$, ... are not slowly varying. In fact $\varphi_1^{(1)}$, $\varphi_2^{(1)}$ become infinite as $O(x^{-(\sigma+1)})$, while proper choice of λ_1 satisfying Eq. (2.49) cancels the singularity in $\varphi^{(1)}$. Thus $\varphi^{(1)}$ remains finite at $x = 0$. It should be true to $\varphi^{(i)}$ for all i since $u(0, f) = 0$ (Eq. (2.36)). The fact may be used to determine λ_i, as follows

$$\lim_{x \to 0} (f_1^{(i)} + \lambda_i f_2^{(i)}) = 0 , \tag{2.50}$$

or

$$\lim_{x \to 0} (\varphi_1^{(i)} + \lambda_i \varphi_2^{(i)}) = \text{finite} .$$

In fact the value of λ_1 determined in this way coincides with the value above in Table II. In actual procedure, $\varphi_1^{(1)}$, $\varphi_2^{(1)}$, $\psi_1^{(1)}$, $\psi_2^{(1)}$ are expanded in series of x near $x = 0$ and their coefficients are evaluated by fitting these expansion solutions to the numerical one from $x = 1$. After these coefficients are determined, λ_1 is determined to cancel the singularity. In any case, it is not convenient to find $\varphi_1^{(1)}$, $\varphi_1^{(2)}$ which become bigger as $x \to 0$ because of the singularity.

An alternative procedure of integration, by means of which the difficulty may be avoided is as follows; Introduce the notation,

$$\mathbf{X} = \begin{pmatrix} \varphi^{(1)} \\ \psi^{(1)} \end{pmatrix}, \quad A = \begin{pmatrix} \frac{1}{\gamma} P_1 + (\gamma - 1) P_5 & \frac{1}{\gamma} P_2 + P_5 \\ P_1 & P_2 \end{pmatrix}, \quad \mathbf{a} = \begin{pmatrix} \frac{1}{\gamma} P_3 \\ P_3 \end{pmatrix}$$

$$\mathbf{b} = \begin{pmatrix} \frac{1}{\gamma} P_4 - P_5 \\ P_4 \end{pmatrix} , \quad \mathbf{m} = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} .$$

Equations (2.44) and (2.45) are then conveniently written in the following vector form,

$$\mathbf{X}' = A\mathbf{X} + \mathbf{a} + \mathbf{b}\lambda_1 \ , \tag{2.51}$$

$$\int_0^1 \mathfrak{m}^* \mathbf{X} x^{\alpha} dx = \frac{1}{(\alpha+1)(\gamma-1)} - I_{01} + \lambda_1(J_0 - I_{02}) \ . \tag{2.52}$$

Now we introduce new functions $\mathbf{Y}$ as,

$$\mathbf{Y} = \begin{pmatrix} \bar{\varphi} \\ \bar{\psi} \end{pmatrix} , \quad \bar{\varphi} \equiv \bar{\varphi}(x), \quad \bar{\psi} \equiv \bar{\psi}(x) \ ,$$

and multiply Eq. (2.51) by $\mathbf{Y}^*$ from the left and integrate it from zero to one to obtain

$$\int_0^1 \mathbf{Y}^* \mathbf{X}' dx = \int_0^1 \mathbf{Y}^* A\mathbf{X} \ dx + \int_0^1 \mathbf{Y}^* (\mathbf{a} + \lambda_1 \mathbf{b}) dx \ ,$$

which yields after an integration by parts,

$$\int_0^1 (\mathbf{Y}^{*\prime} + \mathbf{Y}^* A)\mathbf{X} \ dx = [\mathbf{Y}^* \mathbf{X}]_0^1 - \int_0^1 \mathbf{Y}^* (\mathbf{a} + \lambda_1 \mathbf{b}) dx \ . \tag{2.53}$$

Suppose that we chose $\mathbf{Y}$, so far undetermined, to satisfy the equation,

$$\mathbf{Y}^{*\prime} + \mathbf{Y}^* A = \mathfrak{m}^* x^{\alpha} \ , \tag{2.54}$$

Eq. (2.53), by use of Eq. (2.52), becomes

$$\frac{1}{(\alpha+1)(\gamma-1)} - I_{01} + \lambda_1(J_0 - I_{02}) = [\mathbf{Y}^* \mathbf{X}]_0^1 - \int_0^1 \mathbf{Y}^* (\mathbf{a} + \lambda_1 \mathbf{b}) dx \ . \tag{2.55}$$

Now we choose the value of $\mathbf{Y}$ at $x = 0$ as

$$[\mathbf{Y}^* \mathbf{X}]_{x=0} = 0$$

which is in detail

$$(\bar{\varphi}\varphi^{(1)} + \bar{\psi}\psi^{(1)})_{x=0} = 0 \ .$$

Since $\varphi^{(1)}$, $\psi^{(1)}$ are supposed to be finite at $x = 0$, we may simply put

$$\bar{\varphi}(0) = \bar{\psi}(0) = 0 \quad . \tag{2.56}$$

to satisfy the condition.

Equation (2.54) does not include λ_1 and may be integrated in a straightforward way from the initial values (2.56) at $x = 0$. The solution Υ thus obtained is used to estimate the integral and the value of $[\Upsilon^* X]_{x=1}$ in Eq. (2.55), from which λ_1 is determined as,

$$\lambda_1 = \{[\Upsilon^* X]_{x=1} - \int_0^1 \Upsilon^* a dx + I_{01} - \frac{1}{(\alpha+1)(\gamma-1)}\} (J_0 - J_{02} + \int_0^1 \Upsilon^* b dx)^{-1} \quad .$$

Once λ_1 is evaluated, Eq. (2.51) is now integrated from the initial values in Eq. (2.41) at $x = 1$. The process seems to be much easier than that for Eqs. (2.47), (2.48), since $\varphi^{(1)}$, $\psi^{(1)}$ are expected to vary little in the whole range of $0 \leq x \leq 1$. The process is essentially the same to find λ_2, λ_3, ... as well as $\varphi^{(2)}$, $\psi^{(2)}$, ... in the further approximations.

2.4 Approximations valid in the whole range of y

Although the solution in a series in y may be good near $y \sim 0$, it is certainly not valid near $y \sim 1$. It is also noticed that the expression for the y -R relation given in Eq. (2.22) truncated at a certain power of y, becomes singular at a particular value of y. In the second approximation, $R \to \infty$ at $y = -1/\lambda_1$, where R is supposed to be still finite. In the slightly different expansion procedure of expanding $1/\lambda$ instead of λ (Korobeinikov et al., 1961), namely

$$(\alpha + 1)/\lambda = 1 - \lambda_1 y + \lambda_2' y^2 + \dots \quad ,$$

the y - R relation are obtained from Eqs. (2.16), (2.22) as,

$$R^{\alpha+1} = (R_0^{\alpha+1}/J_0)\, y \exp\{-\lambda_1 y + \frac{1}{2}\lambda_2' y^2 + \dots\} ,$$

or (2.57)

$$y\left(\frac{R_0}{R}\right)^{\alpha+1} = J_0 \exp\{\lambda_1 y - \frac{1}{2}\lambda_2' y^2 + \dots\} ,$$

where the singularity at $y = -1/\lambda_1$ for the second approximation above may be avoided, since an exponential factor remains positive all the time. Nevertheless, it can not be valid near $y \sim 1$, where R should be infinite, while R in Eq. (2.57) terminated after a finite number of terms remains finite at $y = 1$.

An attempt to find an approximate solution valid in the whole range of y (Sakurai, 1959) will be given in the following. Let us first make two assumptions;

i) $f \propto x$ and ii) $\lambda y\, ^{\partial h}/_{\partial y}$ is negligible in the second equation of Eq. (2.15),

$$(f-x)\frac{\partial h}{\partial x} + \lambda y \frac{\partial h}{\partial y} = -h\left(\frac{\partial f}{\partial x} + \frac{\alpha f}{x}\right) .$$

The assumption (i) may be seen in Figure 9 below in Section 3.1.3, where $u/U = f$ for the second approximation for $\alpha = 2$, $\gamma = 1.4$ is shown for various values of y and all curves are actually very similar to straight lines through the origin $x = 0$. The second assumption is based on the following facts: the term becomes small near $y = 0$, since y enters in as a factor, it is also small at $y \sim 1$ where λ becomes small. Although it is not clear that the term remains small in the intermediate values of y, and both assumptions are based on rather vague reasons, they simplify the whole situation remarkably and thus make it possible to provide an approximate formula valid in the whole range of y.

From the first assumption we get simply

$$f = f_0 x , \tag{2.58}$$

where f_0 is the function of y only and given by the condition (2.12) at $x = 1$,

$$f_0 = \frac{2}{\gamma+1}(1-y) \quad . \tag{2.59}$$

By use of Eq. (2.58), the second equation of Eq. (2.15), neglecting the term $\lambda y \, {}^{\partial h}/\partial y$ according to the assumption (ii) becomes

$$\frac{1}{h}\frac{\partial h}{\partial x} = \frac{(\alpha+1) f_0}{1 - f_0}\frac{1}{x} \, ,$$

which is readily integrated to give

$$h = h_0 x^m \quad , \tag{2.60}$$

where we have used the condition (2.12) and put

$$h_0 = \frac{\gamma+1}{\gamma-1}\left(1-\frac{2}{\gamma-1}y\right)^{-1}, \quad m = \frac{(\alpha+1) f_0}{1 - f_0} \quad . \tag{2.61}$$

With use of f, h given in Eqs. (2.58), (2.60), the remaining function g may be determined from the first equation of Eq. (2.15), which is reduced to

$$\frac{\partial g}{\partial x} = \gamma h_0 \left\{f_0\left(1 + \frac{1}{2}\lambda - f_0\right) + \frac{2}{\gamma+1}\lambda y\right\} x^{m+1} \quad ,$$

and integrated to give

$$g = A(x^{m+2} - 1) + g_0 \quad , \tag{2.62}$$

where the condition (2.12) is again used and A, g_0 are given by

$$A = \frac{\gamma h_0}{m+2}\left\{f_0\left(1 + \frac{\lambda}{2} - f_0\right) + \frac{2}{\gamma+1}\lambda y\right\}, \quad g_0 = \frac{2\gamma}{\gamma+1} - \frac{\gamma-1}{\gamma+1} y \quad . \tag{2.63}$$

Since f, g, h are all given simply as power functions of x, the integration of J given in Eq. (2.18) is readily performed and we get

$$J = \frac{1}{m+3+\alpha}\left(\frac{\gamma}{2} h_0 f_0^2 + \frac{A}{\gamma-1}\right) + \frac{g_0 - A}{(\gamma-1)(\alpha+1)} \quad ,$$

which is more conveniently written in the following form:

$$J = P + \lambda Q \quad , \tag{2.64}$$

where P, Q are known functions given as

$$P = \frac{\gamma h_0 f_0}{m+3+\alpha}\left\{\frac{1}{2}f_0 + \frac{1-f_0}{(m+2)(\gamma-1)}\right\} + \frac{1}{(\gamma-1)(\alpha+1)}\left\{g_0 - \frac{\gamma h_0 f_0}{m+2}(1-f_0)\right\},$$

$$Q = \frac{\gamma h_0}{(\gamma-1)(m+2)}\left(\frac{1}{m+\alpha+3} - \frac{1}{\alpha+1}\right)\left(\frac{1}{2}f_0 + \frac{2}{\gamma+1}y\right) \quad . \tag{2.65}$$

Equation (2.64) is used to eliminate J from Eq. (2.18),

$$\lambda = \left[(\alpha+1)J - \frac{y}{\gamma-1}\right]\left[J - y\frac{dJ}{dy}\right]^{-1} ,$$

to obtain a differential equation to determine λ . Practically it is easier to eliminate λ, leading an equation for J ,

$$\frac{J-P}{Q} = \left[(\alpha+1)J - \frac{\gamma}{\gamma-1}\right]\left[J - y\frac{dJ}{dy}\right]^{-1} . \tag{2.66}$$

It will be seen from Eq. (2.66) that J tends to $(\alpha+1)^{-1}(\gamma-1)^{-1}$ when y approaches 1 and accordingly $\lambda \to 0$ there. Since f_0, $m \to 0$; g_0, $h_0 \to 1$ as $y \to 1$ (c.f. Eqs. (2.59), (2.61), (2.63)), we get, from Eq. (2.65),

$$P \to \frac{1}{(\alpha+1)(\gamma-1)}, \quad Q \to \left(\frac{1}{\alpha+3} - \frac{1}{\alpha+1}\right)\frac{\gamma}{\gamma^2-1} \quad \text{as } y \to 1 ,$$

from which we get $J \to (\alpha+1)^{-1}(\gamma-1)^{-1}$ in Eq. (2.66), thus λ is guaranteed to become zero at $y = 1$. Equation (2.66) may be integrated numerically starting from $y = 0$, where $\lambda = \alpha + 1$ and $J = J_0$ given by Eqs. (2.64), (2.65). Equation (2.66) is, however, singular at $y = 0$, since

$$y\frac{dJ}{dy} = J\left(1 - \frac{\alpha+1}{\lambda}\right) + \frac{y}{\lambda(\gamma-1)} ,$$

and $J\left(1-\frac{\alpha+1}{\lambda}\right) \to 0$ as $y \to 0$. To proceed with the numerical integration, it is then necessary to find the values of $(dJ/dy)_{y=0}$ or $(d\lambda/dy)_{y=0}$ which are related in the following way:

$$\left(\frac{dI}{dy}\right)_{y=0} = \left[\frac{d}{dy}\{P+(\alpha+1)Q\}\right]_{y=0} + \left(\frac{d\lambda}{dy}\right)_{y=0} \cdot \left(Q\right)_{y=0} . \tag{2.67}$$

Using this relation as well as Eq. (2.66), it is found

$$\left(\frac{d\lambda}{dy}\right)_{y=0} = (\alpha+1)\{\left[\frac{d}{dy}(P+(\alpha+1)Q)\right]_{y=0} - \frac{1}{(\alpha+1)(\gamma-1)}\}/(P)_{y=0} .$$

It is interesting to note the value of $(\alpha+1)^{-1}(d\lambda/dy)_{y=0}$, which corresponds to λ_1 in Eq. (2.23). For $\gamma = 1.4$ this gives -2.32, -1.82, -1.61 for $\alpha = 0, 1, 2$ respectively, and these should be compared with their exact values -2.138, -1.989, -1.918 given in Section 2.3.3.

With these values given by Eq. (2.67) as well as $(J)_{y=0} = \alpha+1$ at $y = 0$, Eq. (2.66) was integrated numerically for two cases of $\alpha = 1, 2$ and λ, obtained finally from Eq. (2.64) is plotted in Figure 6, where λ-curves given in

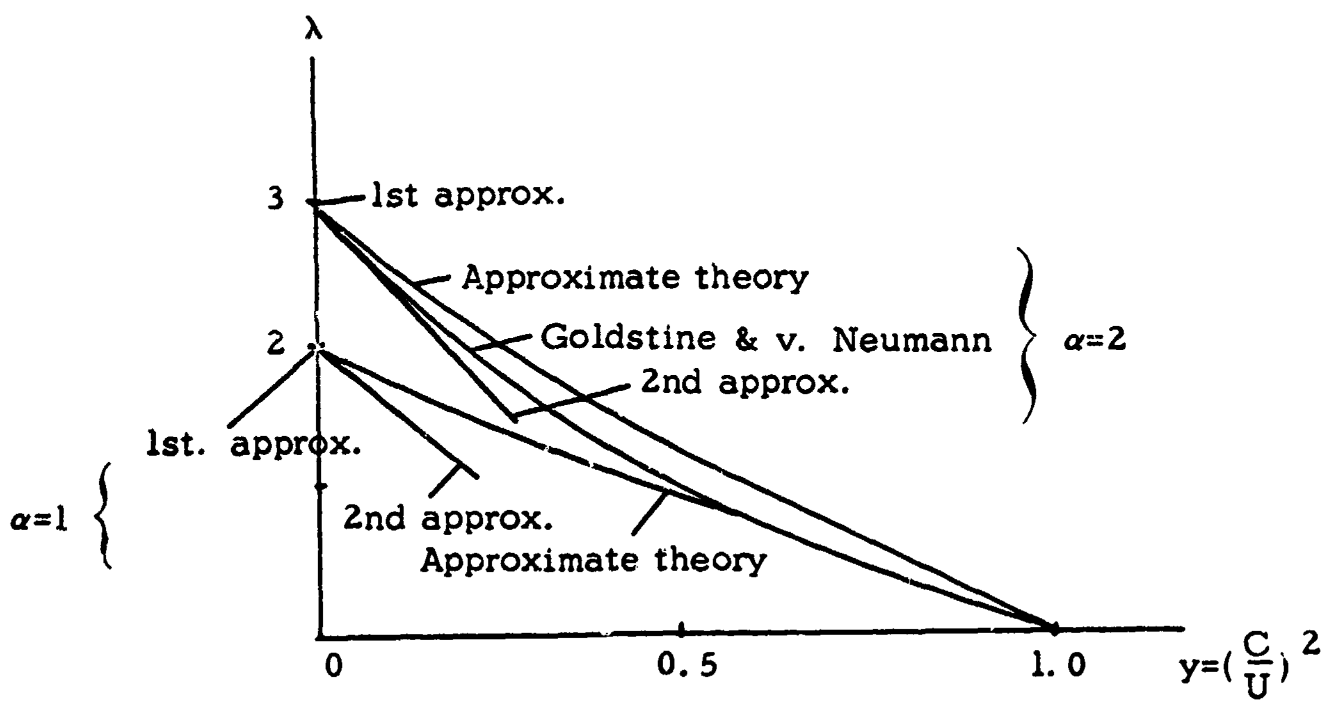

Figure 6

Decay curves (λ vs y) from various results for $\alpha = 1, 2$.

series of y determined in the proceeding section in their first and second approximations, and the curve obtained purely numerically by Goldstine and von Neumann (1950) for $\alpha = 2$ are shown for comparison. Knowing J as a function of y, the R-y relation (or U-R relation) is given directly by Eq. (2.13),

$$y\left(\frac{R_0}{R}\right)^{\alpha+1} = J - \frac{y}{(\alpha+1)(\gamma-1)}, \quad y = \frac{C^2}{U^2},$$

or (2.68)

$$\frac{R}{R_0} = \left(\frac{U^2}{C^2} J - \frac{1}{(\alpha+1)(\gamma-1)}\right)^{-1/(\alpha+1)} .$$

It is easy to see in this expression that R becomes infinite as y approaches 1 because J goes to $(\alpha+1)^{-1}(\gamma-1)^{-1}$ there as mentioned above. Its asymptotic behavior near $y = 1$ is, however, a little different from the exact one, which is known as,

$$1 - y \propto \begin{cases} R^{-1/2} & \text{for } \alpha = 0 \\ R^{-3/4} & \text{for } \alpha = 1 \\ (\log R)^{1/2} R^{-1} & \text{for } \alpha = 2 , \end{cases}$$

(Whitham, 1950, in Sedov, 1957).

It is noted that the expression (2.68) itself is exact as far as J is exact, but J in this approximation is not precise enough at $y \sim 1$ and $1 - y$ behaves as $R^{-(\alpha+1)}$ instead. Nevertheless, Eq. (2.68) with this value of J is expected as a whole to give a rather good approximation to the whole range in U-R relation. Its defect at $y \sim 1$ may be improved in a similar way to that given in Korobeinikov et. al., (1961) by modifying J locally at $y \sim 1$ to fit the exact asymptotic behavior there.

In spite of their comparatively simple forms, the approximate solutions of

f, g, h given by Eqs. (2.58), (2.60) and (2.62) seem to represent the feature fairly well (Sakurai, 1959). Because of its simplicity, the g function in Eq. (2.62) was used to analyze the wave form of thunder, which needed transformation too difficult to perform with the more elaborate formula (Thome, 1962) (See, Section 3.6). This approximate solution has also been used by Jeanmaire (1963) to study the flow field in T-shock-tube. Another approximation theory is considered by Ōsima (1960). He introduced a concept of "quasi-similarity" based on the insensitiveness of the functions f, g, h with the variable y and assumed

$$\frac{\partial f}{\partial y} = \frac{1}{f_0}\frac{df_0}{dy} f, \quad \frac{\partial g}{\partial y} = \frac{1}{g_0}\frac{dg_0}{dy} g, \quad \frac{\partial h}{\partial y} = \frac{1}{h_0}\frac{dh_0}{dy} h ,$$

where f_0, g_0, h_0 are given in Eqs. (2.59), (2.63), and (2.61). Using the expressions, the fundamental system of equation (2.15) are reduced to a system of ordinary differential equations where y comes in as a parameter. Using the above assumption again, dJ/dy is reduced to an integration including f, g, h where y enters as a parameter. The procedure to find the solution for any shock strength y is as follows: start with a guess to λ and integrate the system of differential equations numerically with given value of y, the solution is used to evaluate J, dJ/dy in Eq. (2.17) to get a new λ value, with which we repeat the same process until we get the same value as the proceeding λ value. Actual computations were carried out for $\alpha = 1$ (cylindrical wave) at respective values of $y^{-1/2} \equiv U/C = 1.1, 1.2, 1.4, 1.6, 2, 3, \infty$. The last two cases have been found to coincide with the solution in series of y given in Section 2.3. The solutions were used to compare with his experimental data on the exploding wire (c.f.) Section 3.3).

§3. Application to Various Problems

3.1 Blast Wave

Although application of the foregoing theory to actual blast waves should be the main subject of this article, it is not easy to give a comprehensive survey of it with full use of experimental data. An excellent description of the phenomenon is given in the book "The Effects of Nuclear Weapons" (Glasstone, ed.) (1962) and a comparison of the theory with some data is well reviewed in the book by Korobeinikov et al., (1961). Only a brief description will be given here, by displaying the relations which might be useful for making a comparison with experimental data.

3.1.1 Characteristic lengths, Scaling laws

As long as we assume the point source model, represented by the condition (2.8), the only characteristic length that appears in the entire formulation is R_0 given by Eq. (2.14), and the features corresponding to the different R_0 are simply obtained by scaling. Since R_0 is proportional to $E_\alpha^{1/(\alpha+1)}$ and E_2 to the spherical case is roughly proportional to the weight W of the explosive (of the same kind, of course), the scaling factor is expected to be proportional to $W^{1/3}$. This is usually known as the Scaling law for blast waves and is known to hold for fairly wide range of W (or E_2). (See for example, "Effects of Nuclear Weapons" (1962) pp. 127-146).

It is noticed, however, that R_0 is also related to p_0 in the form $R_0 \propto p_0^{-1/(\alpha+1)}$, and there exists another scaling law concerning the ambient pressure p_0. Since the rate of explosion energy converting into blast wave energy depends significantly on the ambient pressure p_0 (the rate usually

decreases as the pressure is reduced), this scaling law can not be so accurate as the last one above, but nevertheless we may expect the scale of the blast wave from the same source to be magnified as the ambient pressure is reduced. This fact may be useful in producing a situation equivalent to a strong explosion in a laboratory experiment with a relatively small explosion, where the ambient pressure is reduced. The technique was used successfully in the study of a blast wave from an exploding wire (See Section 3.4 below).

It is stressed that the role of R_0 is not merely the scaling factor, but characterizes the entire phenomenon and much information can be obtained by simply estimating the value of R_0 by the formula (2.14). The length R_0 itself indicates a distance in which U/C falls to roughly about 1.8 for $\alpha = 2$ (See Figure 7 below) and this corresponds to about 2.6 of the value of the over-pressure ratio at the shock front (defined by $(p - p_0)_{Y=R}/p_0$, see the shock condition in Eq. (2.6)). These figures are useful when estimating a rough picture of the range in which the individual blast wave is effective.

Apart from the ideal model of a point explosion, there are many other characteristic lengths in actual circumstances, such as the dimensions of the explosive and so on, and some of them are in fact important especially for understanding some details of the phenomenon, while the length R_0 gives an over all feature. Another characteristic length not directly related to the explosion but important for applying the point explosion theory to the actual situation is a distance beyond which the point source assumption is valid. It is suggested by Taylor (1950) for the spherical case that the distance should be much bigger than the radius $\bar{R}$ (say) of a sphere of ambient air equal in mass to that of the explosive.

The criterion should be extended also to other the cases of cylindrical and plane waves.

3.1.2 Decay of the Blast Wave

The decaying characteristics of a blast wave are usually given by the relation between the shock velocity U and the distance R such as illustrated in Figure 1. With use of the values J_0, λ_1, λ_2, ... given in Section 2.3.3, the relation between the velocity U and the distance R (U-R relation) is given by Eq. (2.30). In Figure 7, U-R relations for $\gamma = 1.4$ are shown to the

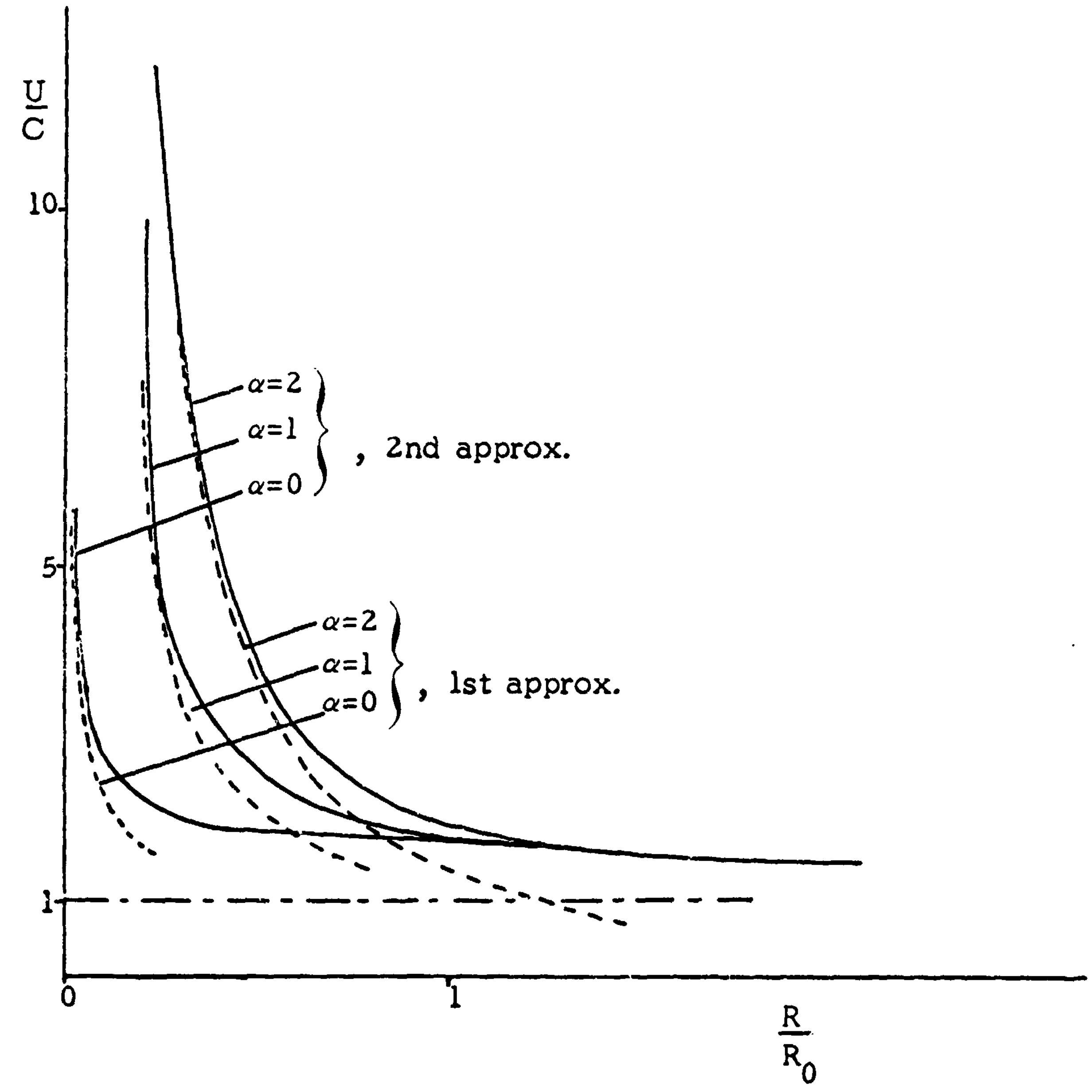

Figure 7
Velocity - distance curves from Eq. (2.30)

second approximation. For $\alpha = 1$, $\gamma = 1.4$, we have the third approximation utilizing the value of $\lambda_2 = 2.7373$ given by Swigart (1960) (See Section 2.3.3), which was used to improve the formula by Lees and Kubota (1957) for the hypersonic flow past a blunt-nosed axisymmetric body.

Sometimes it is more convenient to consider the relation between the peak pressure P (the pressure at the shock front $x = 1$) and the distance R ($P-R$ relation), which can be derived by eliminating U (or y) from Eqs. (2.30) and (2.6) to give

$$\left(\frac{R_0}{R}\right)^{\alpha+1} = J_0\left[\frac{\gamma-1}{2\gamma} + \frac{\gamma+1}{2\gamma}\frac{P}{p_0} + \lambda_1 + \lambda_2\frac{2\gamma}{\gamma+1}\left(\frac{\gamma-1}{\gamma+1} + \frac{P}{p_0}\right)^{-1} + \ldots\right] .$$

The formula in the second approximation for $\gamma = 1.4$ becomes

$$\frac{P}{p_0} = \begin{cases} 1.96\left(\dfrac{R_0}{R}\right)^3 + 2.07 & \text{for } \alpha = 2 , \\ 1.33\left(\dfrac{R_0}{R}\right)^2 + 2.16 & \text{for } \alpha = 1 , \\ 0.69\left(\dfrac{R_0}{R}\right) + 2.33 & \text{for } \alpha = 0 . \end{cases} \tag{3.1}$$

Equations(3.1) have been compared favorably with some experiemental data (Sakurai, 1954).

A P-R curve for $\alpha = 2$ valid for a wider range established both by numerical as well as experimental data may be found in books such as "The Effects of Nuclear Weapons" (1962). Peak pressure P goes down very quickly in the beginning for small R up to about $R = R_0$, where $(P/p_0) - 1$ is 2.6 as noticed above. Beyond that, the rate of decrease slows down and a long distance is needed for the wave to attain accoustic characteristics. In fact, the over-pressure ratio goes down to $O(10^{-1})$ when $R = 10R_0$ approximately and the value of R/R_0 reducing the over-pressure ratio to $O(10^{-2})$ may well be more than 50 . It

is noticed that the over-pressure ratio for the ordinary sound wave is usually of the order of 10^{-6}, in comparison with which the above figure of 10^{-2} is still very high and enough to give people a kind of feeling of "shock", although it may actually give no destractive effects anymore. The region where people feel "shock" is thus expected to spread over quite a range and the region is also roughly corresponding to the domain where the non-linearity of the phenomenon is dominant.

Another important quantity to express the decay feature is λ given by Eq. (2.16) as:

$$\lambda = \frac{R}{y}\frac{dy}{dR} = -\frac{2R}{U}\frac{dU}{dR} \quad .$$

It is stressed (Sakurai, 1955b) that λ is rather useful for both experimental and theoretical studies of the phenomenon, because the characteristics of different kinds of blast wave, corresponding to different values of α can be compared in a single diagram as shown in Figure 6, where numerical as well as approximate results for $\alpha = 2$ are compared with some experimental data.

For a more direct way of plotting experiment data, $R - t$ diagram is used, in which the theoretical curve can be obtained by integrating Eq. (2.30). The relation in the second approximation becomes

$$\frac{ct}{R_0} = \sqrt{J_0}\,K, \quad K = \int_0^{R/R_0} \{z^{-(\alpha+1)} - J_0\lambda_1\}^{-1/2}\,dz \quad . \tag{3.2}$$

K for $\alpha = 0$, 1 are found explicitly as,

$$K = \begin{cases} (-J_0\lambda_1)^{-3/2}\{\sqrt{v(1+v)} - \log(\sqrt{1+v} + \sqrt{v})\}, \quad v = -J_0\lambda_1\dfrac{R}{R_0}, & \text{for } \alpha = 0 \;, \\[2ex] (-J_0\lambda_1)^{-1}\left\{\sqrt{1 - J_0\lambda_1\left(\dfrac{R}{R_0}\right)^2} - 1\right\}, & \text{for } \alpha = 1 \;. \end{cases}$$

K in the first approximation is reduced to

$$K = \frac{2}{\alpha+3}\left(\frac{R}{R_0}\right)^{(\alpha+3)/2},$$

and the formula for $\alpha = 2$ shows a good agreement with the data for an atomic explosion in the range of $R = 20 \sim 185$m. (Taylor, 1950).

Ōshima (1960, 1962) thoroughly checked the relation (3.2) in the case $\alpha = 1$, $\gamma = 1.4$ with his experimental data on blast waves from wire explosion (c.f. Section 3.3 below). Figure 8 (Ōshima, 1960) shows the range where the

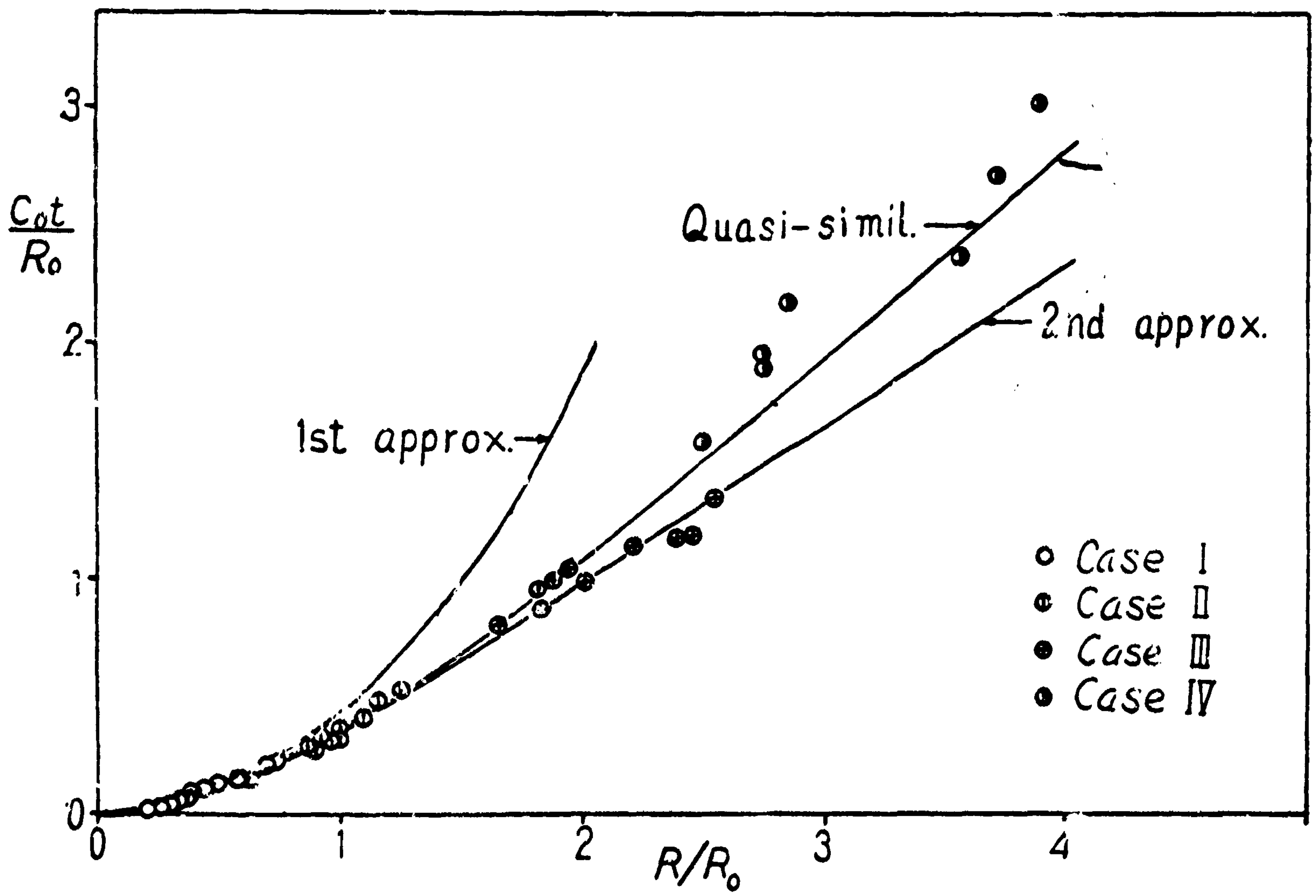

Figure 8

Experimental data for shock arrival time compared with various theoretical curves for $\alpha = 1$, $\gamma = 1.4$ (Ōshima, 1960)

formula is valid, displaying his approximate solution mentioned in Section 2.4.

3.1.3 Flow Field inside Blast Wave

The velocity, the pressure and the density inside the blast wave are represented by f, g, h given in Eq. (2.11) and their changing behavior with respect to time can be seen in their graphs against x for various fixed values of y (or U/c). In Figures 9, 10, and 11, these graphs for $\alpha = 2$ are shown for various values of U/c . Those corresponding to other cases of $\alpha = 0, 1$ are very similar in nature to the graphs above.

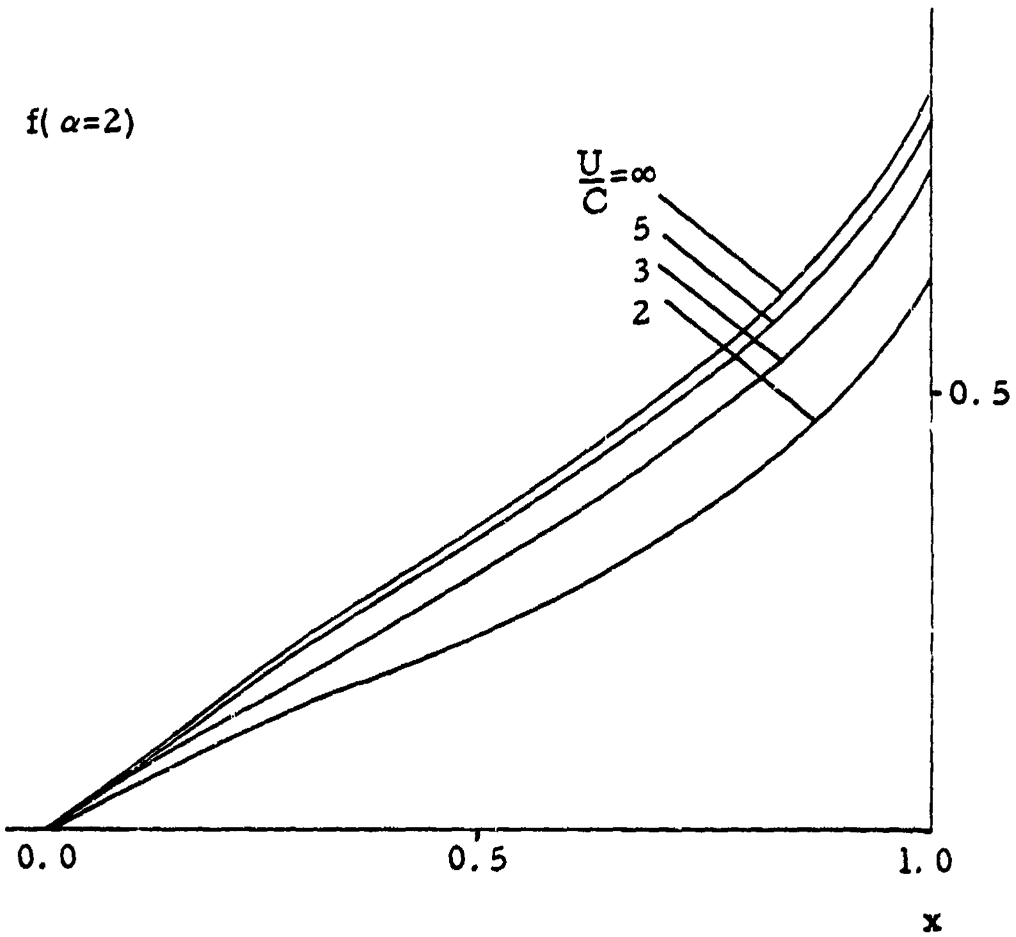

Figure 9

f(x, y) for various values of $y^{-1/2}$ (≡ U/c)

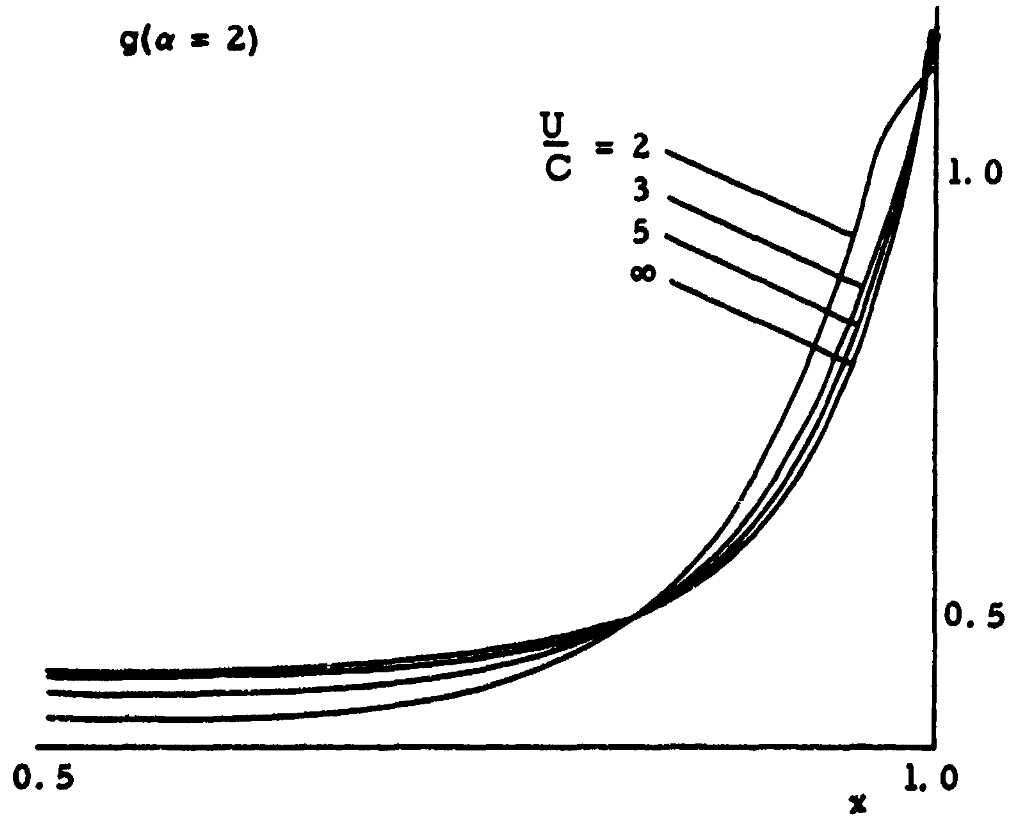

Figure 10

$g(x, y)$ for various values of $y^{-1/2}$ ($\equiv U/c$)

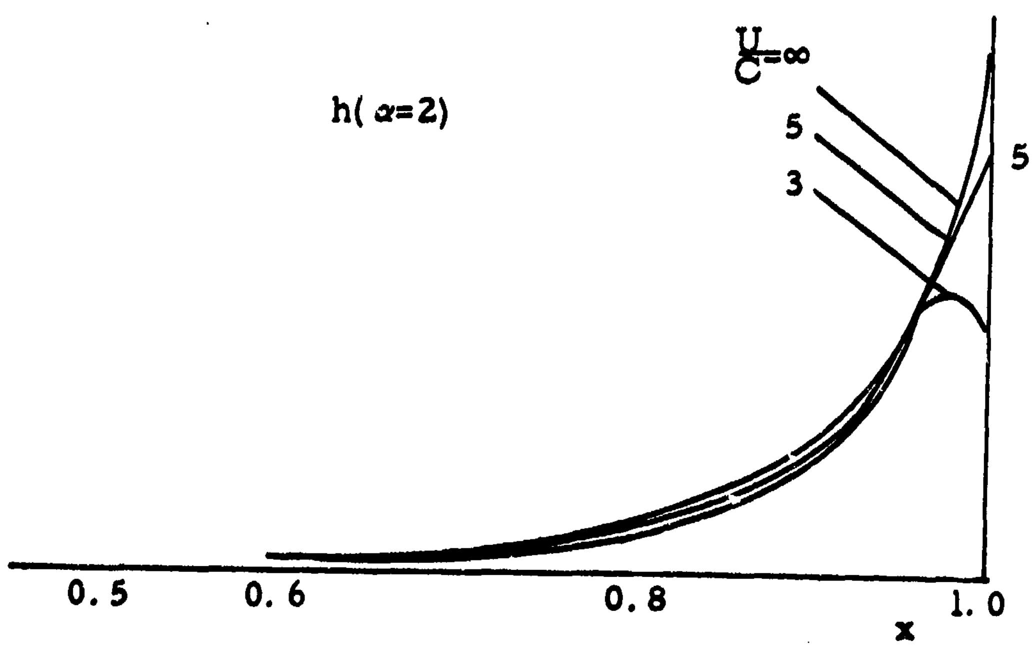

Figure 11

$h(x, y)$ for various values of $y^{-1/2}$ ($\equiv U/c$)

It is noted that the usual means of measurements of the pressure, in actual observation, is performed at a fixed point and expressed as a function of time, while the above curve of g is concerned with a fixed time. Sometimes it is necessary to convert the above curves to the one corresponding to the fixed point. (An example of the conversion is given in Section 3.7).

3.2 Non-uniform Field

It is sometimes necessary to take into account the non-uniformity of the medium, through which the blast wave is propagating. Blast wave propagation in a star is a typical example of this case, where the initial ambient pressure and density p_0, ρ_0 ahead of the shock wave can not be assumed constant, but are non-uniform because of gravitational effects. Even in the case of the terrestrial scale, for huge blast waves such as those produced by nuclear explosions it is necessary to take into account the fall in pressure and density in the atmosphere as the altitude is increased.

It is possible to modify the blast wave theory for a non-uniform medium by simply assuming that the initial density (and the pressure as well) may be expanded in a power series of R as

$$\rho_0 = a_0 + a_1 R + \dots ,$$

where $a_i (i = 0, 1, 2, \dots)$ are constants assumed to be known. A series expansion solution similar to that developed in Section 2.3 is then modified to incorporate the non-uniformity effects represented by a_1, a_2, ... with additional terms in the series.

The approach will be illustrated below in the case of a spherical shock wave in a star (Sakurai, 1956). Suppose the initial density distribution in star is

given by the following Emden's equation (slightly modified from the original one, see for example, Chandrasekhar, 1939):

$$\frac{d}{dz}\left(z^2 D^{\gamma-2}\frac{dD}{dz}\right) + 6A^2 z^2 D = 0 \quad , \tag{3.3}$$

where we put

$$D = \frac{\rho_0}{\rho_c}, \quad z = \frac{R}{R_0}, \quad A = \frac{R_0}{L}, \quad R_0 = \left(\frac{E}{4\pi p_0}\right)^{1/3} \quad ,$$

p_c, ρ_c are the values of p_0, ρ_0 at the center of the star, and L represents a length connected with the dimensions of the star given by $L = \left(\frac{3\gamma p_c}{2\pi\rho_c^2 G}\right)^{1/2}$, while G is the constant of gravitation. We find from Eq. (3.3),

$$D = 1 - A^2 z^2 + \frac{13-5\gamma}{10} A^4 z^4 + \dots \quad . \tag{3.4}$$

It is also necessary to modify the fundamental system of equations to include the terms from the gravitational effects. Thus Eq. (2.1) is modified to give:

$$\frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial r} - \frac{Gm}{r^2} \, ,$$

supplemented by

$$\frac{\partial m}{\partial r} = 4\pi r^2 \rho \quad ,$$

while Eqs. (2.2), (2.4) are unchanged and

$$\begin{aligned} \frac{D\rho}{Dt} &= -\rho\left(\frac{\partial u}{\partial r} + \frac{2u}{r}\right) \quad , \\ \frac{Dp}{Dt} &= -\gamma p\left(\frac{\partial u}{\partial r} + \frac{2u}{r}\right) \quad . \end{aligned} \tag{3.5}$$

The condition (2.8) for an instantaneous explosion of energy is also modified to include gravitational energy,

$$\int_0^R \left(\frac{1}{2}u^2 + \frac{1}{\gamma-1}\frac{p}{\rho} - \frac{Gm}{r}\right)\rho 4\pi r^2 dr$$

$$- \int_0^R \left(\frac{1}{\gamma-1}\frac{p_0}{\rho_0} - \frac{Gm_0}{R}\right)\rho_0 4\pi R^2 dR = E \ , \qquad (3.6)$$

where m_0 is defined by

$$m_0 = \int_0^R 4\pi R^2 \rho_0 \, dR \ .$$

The shock conditions (2.6) are the same, c is given by $(\gamma p_0/\rho_0)^{1/2}$ and is not constant in the present case, but a function of R, which will be expressed in series of powers of z using Eq. (3.4). We have another condition at the shock front,

$$(m)_{l=R} = m_0 \ . \qquad (3.7)$$

Now to express the solution in series form, it is not possible to expand it in series of y(or U^{-2}) as given in Section 2.3, since U in the present case is no longer necessarily a monotonically decreasing function of R, but may even increase as a result of decreasing density and pressure in the equilibrium state, and would then cease to be single-valued. Accordingly we can not use y as an independent variable and we use z given in Eq. (3.3) instead. As is noted in Section 2.2, the choice of the variable y is not indispensable, but any other variables insensitive to the solution can be used alternatively. Thus we introduce independent variables x, z defined by

$$\frac{r}{R} = x, \quad \frac{R}{R_0} = z$$

and transform u, p, ρ, m in a similar way as to that applied in Eq. (2.11),

$$\frac{u}{U} = f(x,z), \quad \frac{p}{p_c} = P(z)\left(\frac{U}{C}\right)^2 g(x,z), \quad \frac{\rho}{\rho_c} = D(z)\ h(x,z) \quad ,$$

$$\frac{m}{m_c} = M(z)\ n(x,z) \quad , \tag{3.8}$$

where we put

$$\frac{p_0}{p_c} = P(z), \quad \frac{m_0}{m_c} = M(z), \quad m_c = \frac{4}{3}\pi \rho_c R_0^3 \ .$$

Equations (3.5) are then transformed into,

$$\left\{\begin{aligned}
&(f-x)\frac{\partial f}{\partial x} + z\frac{\partial f}{\partial z} + \frac{z}{U}\frac{dU}{dz} f = -\frac{1}{\gamma h}\frac{\partial g}{\partial x} - 2A^2\frac{C_c^2}{U^2}\frac{M}{z}\frac{n}{x^2} \quad ,\\
&\frac{\partial n}{\partial x} = 3z^2\frac{D}{M}x^2 h \quad ,\\
&(f-x)\frac{1}{h}\frac{\partial h}{\partial x} + \frac{z}{h}\frac{\partial h}{\partial z} + \frac{z}{D}\frac{dD}{dz} = -\frac{\partial f}{\partial x} - \frac{2f}{x} \quad ,\\
&(f-x)\frac{1}{g}\frac{\partial g}{\partial x} + \frac{z}{g}\frac{\partial g}{\partial z} + \frac{z}{D}\frac{dD}{dz} + \frac{2z}{U}\frac{dU}{dz} = -\gamma\left(\frac{\partial f}{\partial x} + \frac{2f}{x}\right) \quad ,
\end{aligned}\right. \tag{3.9}$$

while the condition (3.6) gives,

$$\frac{U^2}{C_c^2} Dz^3 J - 2\gamma A^2 MDz^2 K - I = 1 \ , \tag{3.10}$$

where we have put

$$J = \int_0^1 \left(\frac{\gamma}{2} h f^2 + \frac{g}{\gamma-1}\right) x^2 dx, \quad K = \int_0^1 h\, n x^2\, dx$$

$$I = \int_0^z \left(\frac{P}{\gamma-1} - 2\gamma A^2 \frac{MD}{z}\right) z^2\, dz \ .$$

The conditions (2.6) and (3.7) at the shock front $(r = R)$ go to

$$f(1,z) = \frac{2}{\gamma+1}\left\{1 - \left(\frac{C}{U}\right)^2\right\}, \quad g(1,z) = \frac{2\gamma}{\gamma+1} - \frac{\gamma-1}{\gamma+1}\left(\frac{C}{U}\right)^2 \ ,$$

$$h(1,z) = \frac{\gamma+1}{\gamma-1}\left\{1 + \frac{2}{\gamma-1}\left(\frac{C}{U}\right)^2\right\}^{-1}, \quad n(1,z) = 1 \ . \tag{3.11}$$

We have thus reduced the problem to finding the solution of the system of equations (3.9), (3.10) subject to the boundary condition (3.11). In these equations, such quantities as D, P, M, C, I are assumed to be known in series of z . The expression for D is given in Eq. (3.4), from which expressions for P, M, C, I are derived as

$$
\left\{
\begin{aligned}
&P = 1 - \gamma A^2 z^2 + \frac{4}{5}\gamma A^4 Z^4 + \dots , \\
&M = Z^3(1 - \frac{3}{5}A^2 Z^2 + \frac{39-15\gamma}{70} A^4 Z^4 + \dots) , \\
&\frac{C^2}{C_c^2} = 1 - (\gamma-1)\, A^2 Z^2 + \frac{3}{10}(\gamma-1)\, A^4 Z^4 + \dots , \\
&I = \frac{z^3}{\gamma-1}\{\frac{1}{3} - \frac{2\gamma-1}{5}\gamma A^2 Z^2 + \frac{4\gamma}{35}(4\gamma-3)\, A^4 z^4 + \dots\} .
\end{aligned}
\right.
\qquad (3.12)
$$

The solution of the system will be found in a similar manner to that used in Eq. (2.19), in power series of z . We assume first,

$$
\left\{
\begin{aligned}
&f = f^{(0)} + zf^{(1)} + z^2 f^{(2)} + \dots , \\
&g = g^{(0)} + z\, g^{(1)} + z^2 g^{(2)} + \dots , \\
&h = h^{(0)} + z\, h^{(1)} + z^2 h^{(2)} + \dots , \\
&n = n^{(0)} + z\, n^{(1)} + z^2 n^{(2)} + \dots ,
\end{aligned}
\right.
\qquad (3.13)
$$

where $f^{(i)}$, $g^{(i)}$, $h^{(i)}$, $n^{(i)}$ $(i = 0, 1, \dots)$ are all functions of x to be determined.

Inserting the expression (3.13) in J, K in Eq. (3.10), they are expanded as

$$
J = J_0(1 + \alpha_1 z + \alpha_2 z^2 + \dots),\quad K = K_0(1 + \beta_1 z + \beta_2 z^2 + \dots), \qquad (3.14)
$$

where J_0, α_1, α_2, ...; K_0, β_1, β_2, ... are constants given by integrals similar to J_0, σ_1, σ_2, ... given in Eq. (2.21). With use of Eqs. (3.10), (3.12) and (3.14), U is expressed as

$$\frac{C_c^2}{U^2} = J_0\, z^3 \{1 + \alpha_1 z + (\alpha_2 - A^2) z^2 + (\alpha_3 - \frac{1}{3}\frac{1}{\gamma-1} - \alpha_1 A^2) z^3 + \ldots\} \quad . \tag{3.15}$$

Substituting expressions (3.12), (3.13) and (3.15) in Eq. (3.9), and comparing the same powers of z on both sides, we may get systems of equations for $f^{(i)}$, $g^{(i)}$, $h^{(i)}$, $n^{(i)}$ similar to that given in Eqs. (2.25), (2.26). In the same way we get from the condition (3.11),

$$\begin{cases} f^{(0)}(1) = \frac{2}{\gamma+1},\ g^{(0)}(1) = \frac{2\gamma}{\gamma+1},\ h^{(0)}(1) = \frac{\gamma+1}{\gamma-1},\ n^{(0)}(1) = 1 \ , \\ f^{(1)}(1) = f^{(2)}(1) = g^{(1)}(1) = g^{(2)}(1) = h^{(1)}(1) = h^{(2)}(1) = 0 \ , \\ f^{(3)}(1) = -\frac{2}{\gamma+1} J_0,\ g^{(3)}(1) = -\frac{\gamma-1}{\gamma+1} J_0,\ h^{(3)}(1) = -\frac{2(\gamma+1)}{(\gamma-1)^2} J_0,\ n^{(3)}(1) = 0 \ , \\ \ldots \ . \end{cases} \tag{3.16}$$

Now the first system of equations with its boundary condition given in Eq. (3.16) is the same as the first approximation given in Section 2.3.2 except for the additional equation for $n^{(0)}(x)$,

$$n^{(0)\prime} = 3x^2 h^{(0)} \quad .$$

The second system for $f^{(1)}$, $g^{(1)}$, $h^{(1)}$, $n^{(1)}$, α_1 turns out to give simply,

$$f^{(1)} = g^{(1)} = h^{(1)} = n^{(1)} = \alpha_1 = 0 \quad .$$

Because of this, the following systems are simplified and the fourth system of equations for $f^{(3)}$, $g^{(3)}$, $h^{(3)}$, $n^{(3)}$ and α_3 is reduced to exactly the same as that for the second step to the approximation given in Section 2.3.3 except the equation for $n^{(3)}$. At the same time the third system of equations resulting from the term in z^2 is reduced to a similar but slightly different system, which can be solved, and the numerical procedure to determine

$f^{(2)}(x)$, $g^{(2)}(x)$, $h^{(2)}(x)$ and α_2 is the same as that developed in Section 2.3.3, the value of α_2 so determined for $\gamma = 1.4$ being $0.182\ A^2$. With this value of α_2 as well as the values of J_0, λ_1 for $\gamma = 1.4$ given in Section 2.3, the velocity of front shock U given in Eq. (3.15) becomes,

$$\left(\frac{C_c}{U}\right)^2 = 0.596\ Z^3\ (1 - 0.82\ A^2\ Z^2 - 1.14\ Z^3 \ldots)$$

or

$$\frac{U}{C_c} = 1.30\left(\frac{R}{R_0}\right)^{-3/2}\left\{1 + 0.41 A^2\left(\frac{R}{R_0}\right)^2 + 0.57\left(\frac{R}{R_0}\right)^3 + \ldots\right\}\ .$$

It is readily seen that functions $f^{(2)}$, $g^{(2)}$, $h^{(2)}$, $n^{(2)}$ and the constant α_2 to be determined in the third step are all proportional to A^2, which represents the effects of non-uniformity in the initial distribution given in Eq. (3.4). Thus the second order term in z in the series expansion gives the non-uniformity effect, while the first and the third terms give the ordinary attenuation effect. In the further approximations beyond the fourth step, the two effects are not separated but enter in combination.

Many other problems may be treated in a similar way to that illustrated above as long as the non-uniformity in the medium remains within reasonable bounds, so that it can be well represented by a series expansion. The expansion is not necessarily to be symmetric, (such as above, where it is a function of R only), but may have terms including variables other than R . The solution may be assumed in an expansion form fitting the series expansion to the initial distribution. A good example of the procedure is seen in the problem of a huge terestrial explosion such as that produced by nuclear bombs, in which the effects of the density decrease upwards must be taken into account. The initial density ρ_0 may be assumed in this case as:

$$\rho_0 = a_0 + a_1\zeta + \dots \quad ,$$

where ζ is taken as a vertical coordinate, with $\zeta = 0$ at the center of explosion and the solution is found in series expansion form as

$$\begin{cases} u/\bar{U} = f^{(0)}(x) + \bar{Z}\, f^{(1)}(x,\theta) + \dots \quad , \\ v/\bar{U} = \bar{Z}\, f_v^{(1)}(x,\theta) + \dots \quad , \\ (p/p_0)\left(\frac{\bar{U}}{c}\right)^2 = g^{(0)}(x) + \bar{Z}\, g^{(1)}(x,\theta) + \dots \quad , \\ \rho/\rho_0 = h^{(0)}(x) + \bar{Z}\, h^{(1)}(x,\theta) + \dots \quad , \end{cases}$$

where θ is the angle between the direction ζ and the radius r $(r\cos\theta = \zeta)$ and,

$$\bar{Z} = \frac{\bar{R}}{R_0} \, , \quad \bar{U} = \frac{d\bar{R}}{dt} \quad ,$$

while the shape of the shock front is given by:

$$R = \bar{R}\,(1 + \bar{Z}\,\varphi_1(\theta) + \dots) \quad .$$

More details of the procedure are found in Korobeinikov et. al., (1961) with a slightly different initial distribution of density

$$\rho_0 = a_0 + a_1\,\zeta^n, \quad n\text{: constant} \quad .$$

There is another way of approach to the problem. This is to find the exact solutions of the fundamental equation by seeking a similarity solution, which satisfies the equation for specified initial density, determined from the consistency of the existence of these exact solutions (see Korobeinikov et. al. 1961, also Korobeinikov and Karlikov, 1960). Since these solutions are usually valid at constant Mach number at the shock front, the effects of the changing Mach

number can be obtained by the perturbation method similar to those used above. A simple example of these specified initial distributions is given by $\rho_0 \propto R^{-\omega}$ for certain values of ω depending on γ, which provides a similarity solution for a strong shock wave (See for example, Sedov, 1957). Perturbation from the similarity solution by series expansion in z (or y) has been treated in a similar manner to that used above by several authors (most of the works are in Korobeinikov et. al., 1961; also see Rogers, 1956). Among them, there is an interesting case to which the solution of the second approximation can be found exactly where $\omega = \frac{(3\alpha+1) + \gamma(1-\alpha)}{\gamma+1}$ (Korobeinikov and Ryazanov, 1959). This is actually the generalized version of the special case, $\alpha = 2$, $\gamma = 7$ mentioned in Section 2. 3. 3.

Now the method described is essentially a perturbation procedure starting from a similarity solution and the method can be applied to many other different problems. The problem of the propagation of shock wave produced by <u>non-uniform motion of a piston</u> can be treated in the same way above to find the effects of the non-uniform motion. The flow caused by a uniform piston motion (not necessarily a piston in ordinary sense, but possibly a uniformly expanding cylinder or sphere) can be expressed by a similarity solution and the effects due to the non-uniform pitson motion are incorporated in the perturbation terms similar to those above (c. f. Kochina and Mel'nikova, 1958, 1960). The problem is also related to the hypersonic flow around a blunt nosed slender body. More complicated application of the method is seen in the problem of shock propagation due to non-uniform piston motion in a conducting fluid with a uniform magnetic field. In this problem, two effects, from non-uniform piston motion

and the non-uniform medium appear at the same time, since the interaction of the flow with the magnetic field distorts the symmetry and thus induces an effect similar to that from a non-uniform medium. The problem, though complicated, can be treated by the same method of perturbation as that described above (Ness et. al., 1963).

In dealing with the propagation of shock wave in stars, we always encounter the problem of determining the behavior of the shock wave at the surface of the star, where the density goes to zero and the solution becomes singular, and it is necessary to find a solution fitting this singularity locally (Gandel'man and Frank-Kamenetskii, 1956, Sakurai, 1960).

3.3 Exploding Wire Phenomenon

Shock waves produced by electric wire explosions have been employed by many research workers as a useful tool in the diagnosis of the exploding wire phenomenon or used directly for various practical purposes (c.f. Chace and Moore, "Exploding Wires", Vol. 1, 1959; Vol. 2, 1962; Vol. 3, 1964). Since the exploding wire is required as a line source producing a cylindrical blast wave similar in nature to the one discussed above, attempts have been made to utilize blast wave theory for the purpose of studying the phenomenon (c.f. Section "Shock Waves" in "Exploding Wires" above), which shows different features depending on the purpose of the study. Simple application of the theory seems inadequate (Bennett, 1959) and only give rise to confusion, suggesting more careful consideration of the situation. It will be shown below that there is a proper region for the theory to apply and observation in this range shows good agreement with theoretical results (Ōshima, 1960, 1962). These two

different features will be given in the following.

3.3.1 Applicability of Blast Wave Theory to Exploding Wire Phenomenon

Since the shock wave produced by the exploding wire was noticed to be a useful tool to clarify the exploding wire phenomenon, various measurements have been performed to pick up the shock wave by different experimental techniques (c.f. "Exploding Wires" above). It has been found that measured values of distance and arrival time of the shock wave plotted in an $R^2 - t$ diagram are usually almost on a straight line, in accordance with the result of the theory of the strong cylindrical blast wave (c.f. Eq. (3.2)), and the amount of total energy of the wave estimated from the slope of the line in $R^2 - t$ diagram (using the relation (2.14)) agrees reasonably will with the corresponding value expected from the other estimates (Bennett, 1959b, 1962a, Jones and Gallet, 1962). Nevertheless, it has been shown by Bennett (1959), that there are some serious deviations between the more detailed results of the experimental data and the blast wave solution. Slopes of the line in $R^2 - t$ diagram for various ambient densities under the same energy input do not show inverse square root dependency on the density, as predicted by theory, which gives

$$R^2 = (1/2)(\gamma E_1/J_0 \rho_0)^{1/2} t$$

(Eq. (3.2)), (Ōshima (1960) also noticed that the difference in ambient pressure hardly affects the flow field inside the blast wave in its early stages). Secondly, the careful examination of the data reveals that $R^{5/3}$ vs t rather than R^2 vs t provides a better fit.

It is known that the relation $R^n \propto t$ for n different from 2 holds for a

strong shock wave from a source with energy addition varying with time according to the law $E \propto t^{\frac{4}{n}-2}$ which is usually attained by an expanding piston motion, (c.f. for example, Korobeinikov et. al., 1961). Since there are many reasons to believe that energy is being supplied in the course of growth at disturbance from an exploding wire, an exponent different from 2 might indicate a similarity flow with energy addition. However it is not clear whether the exploding wire behaves as an expanding piston obeying the power law required by the similarity solution (Bennett, 1959b).

Since the phenomenon would appear to be a far more complicated one than that given simply by the line source theory, it seems more appropriate to find the flow field numerically. This was done by Rouse (1959) based on a model reproducing the anticipated physical situation as precisely as possible. The numerical results were compared with the experimental data so that we may justify the postulated physical picture. Although there still exists a limitation in the capacity in the computing machine, it may be possible to carry out such a program for elaborate models of the type considered by Bennett (1962b, 1963).

The above remarks, however, do not necessarily rule out the possibility of the applicability of the blast wave theory to exploding wire phenomenon, provided this is used in the proper region. It is recalled that the blast wave theory should be applied to a region of R where the fundamental assumption of instantaneous line (or point) source is satisfied. The criterion is roughly given by $R \gg \bar{R}$ (c.f. Section 3.1.1) where $\bar{R}$ is the radius of a cylinder of air equal in mass to that of the explosive. Take for example a copper wire of diameter, 0.1mm, which gives the value of $\bar{R}$ as about 0.4 cm; accordingly R should exceed 4 cm or so in this case.

Under these conditions we can hardly expect to find any relation between the experimental data for $R < 4$ cm and the line source theory to this case. Now there is another limitation due to the relation $R^2 \propto t$ resulting from the strong-shock assumption, which neglects $O(y)$ terms and retains only the first approximation (c.f. Section 2.3). Suppose we impose the criterion $y < 0.1$ which corresponds roughly to $R < 0.4\ R_0$. This limitation can be eased by use of the higher approximations. Even with these, however, it is usually convenient to use the region near $R = R_0$ to get better fit because of the nature of the R-t curve. The magnitude of R_0 in the cylindrical case is given as

$$R_0 = \left(\frac{E}{2\pi p_0}\right)^{1/2}$$

where E is the explosion energy per unit length of the line source and is expected to be $5 \sim 100$ joule/cm in the typical experiments on an exploding wire. The values of R_0 for this range of energies are $2.8 \sim 12.6$ cm at $p_0 = 1$ atm.

To find the explosion energy by use of the proportionality relation on the $R^2 - t$ curve, it is essential to use the experimental data in the region for

$$R \gg \bar{R}\ , \quad R \lesssim 0.4\ R_0\ . \tag{3.17}$$

It often happens, however, that when R satisfies the condition $R \gg \bar{R}$ it exceeds $0.4\ R_0$. Therefore the region where the proportionality in $R^2 - t$ holds, may not exist but it should be stressed that this is a situation where the higher approximations should be used to estimate the energy from the data in the region of $R > 0.4\ R_0$. Practically, it may not be easy to find data corresponding to this region, because of limitations in the capacity of ordinary techniques to pick up the shock wave, since these usually utilize the properties prominent

in a stronger shock wave such as its luminosity. There is also a more fundamental difficulty due to the distortion of cylindrical symmetry in the flow field at the distances in question. Nevertheless, the measurements satisfactory for picking up the weaker shock show good agreement with the results of the blast wave theory as long as the experiments are especially designed to retain conditions of symmetry. One of this kind of experiments will be given in the next section.

3.4.2 Cylindrical Blast Wave from Wire Explosion

Ōshima (1960, 1962) performed an extensive experiment specially directed forward the study of the cylindrical blast wave produced by a wire explosion. The experiment was deliberately designed with the intention of studying the details of the blast wave itself. Firstly, wires were exploded between two plates (actually inside the chamber of a thin cylinder), so that the flow field could retain its cylindrical symmetry at any distance from the wire. Secondly, the explosions were executed under reduced pressures (See Table III below) which resulted in attaining larger values for R_0 and made it possible to obtain a region in R satisfying the conditions (3.17).

Another merit of having larger R_0 by reducing the pressure is to have more accuracy in the measurements of the flow field, since the field is in fact magnified with a larger characteristic length. It is noted that a larger R_0 can also be obtained by putting in more electric energy into discharge, but increased amounts of energy would more likely give rise to other troubles. Thirdly, he utilized the Mach-Zehnder interferometer to measure the phenomenon. The technique is familiar in experimental studies of aerodynamics and can

provide information on density variation, which is not required to be as pronounced in other methods of observation such as these making use of luminosity, and the measurements could be carried out over a wider range in R , which covers not only the range where the series expansion solution (Section 2. 3) is effective, but outwards beyond that of the very weak shock wave region. Furthermore, it should be noticed that the measurement supplies the whole knowledge on the density distribution in the entire field within the blast wave, and makes it possible to compare the distribution with the one obtained from blast wave theory, so that more detailed comparison is obtained than in the distance-time relation (R - t diagram). Bennett (1962) also used the interferometer but he utilized it to study the exploding wire phenomenon itself limited to the range of very small R .

Table III

Case	I	II	III	IV	V	VI	VII	VIII
Range of shock Mach No.	6.5~3	2.1~1.5	1.53~1.39	1.13~1.11		1.08~1.12		--
Chamber pressure (mm Hg)	10	40	130	760		760		10
Discharge voltage(kV)	6	6	6	6		4~5		7
Discharge condenser (F)	8	8	8	8		4		8
Wire diameter (mm)	0.10	0.10	0.10	0.10	0.10	0.12	0.15	0.10

Various cases of reduced pressures as well as other conditions in the experiments carried out by Ōshima are shown in Table III.

The energy inputs in the cases of I, II, III, IV, and VIII, are almost the same, but the different values of the corresponding reduced pressures are effective in producing blast waves of different scales. Density distributions

measured by the interferometer in these different cases reveal that there are some distinct differences in the flow features at the respective stages of the blast wave. Some typical data for density distributions at the different stages are reproduced in Figures 12, 13, 14, from Ōshima's report (1960). These should be compared with theoretical h-curves given in Figure 11 in Section 3.1. For strong shock such as in Case I (where the Mach number is larger than about 2 but not so large as to violate the condition $\bar{R} << R$) the flow features as shown in Figure 12 agree closely with the blast wave theory almost entirely

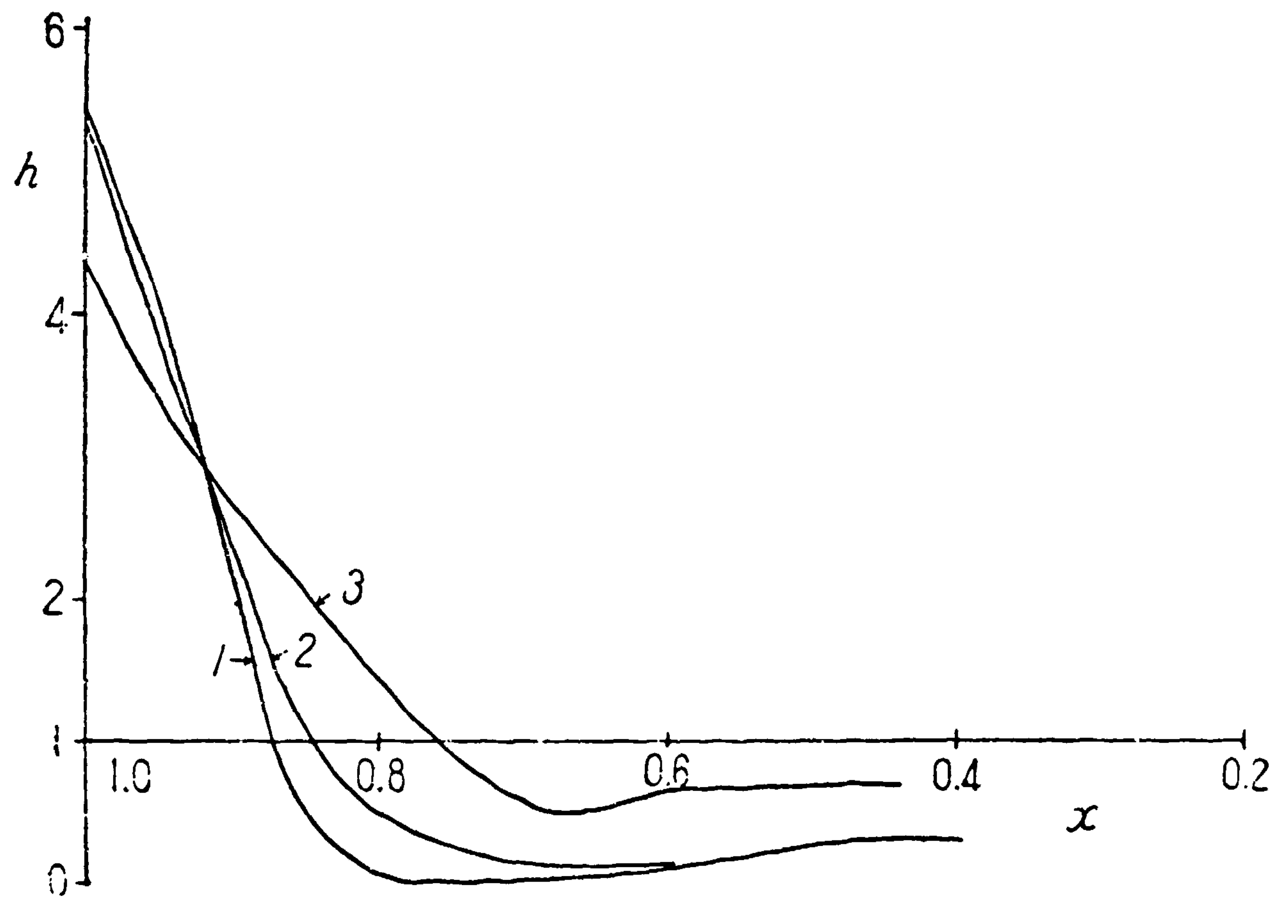

Figure 12

Experimental h vs x distributions for case I-1 (Ōshima, 1960)

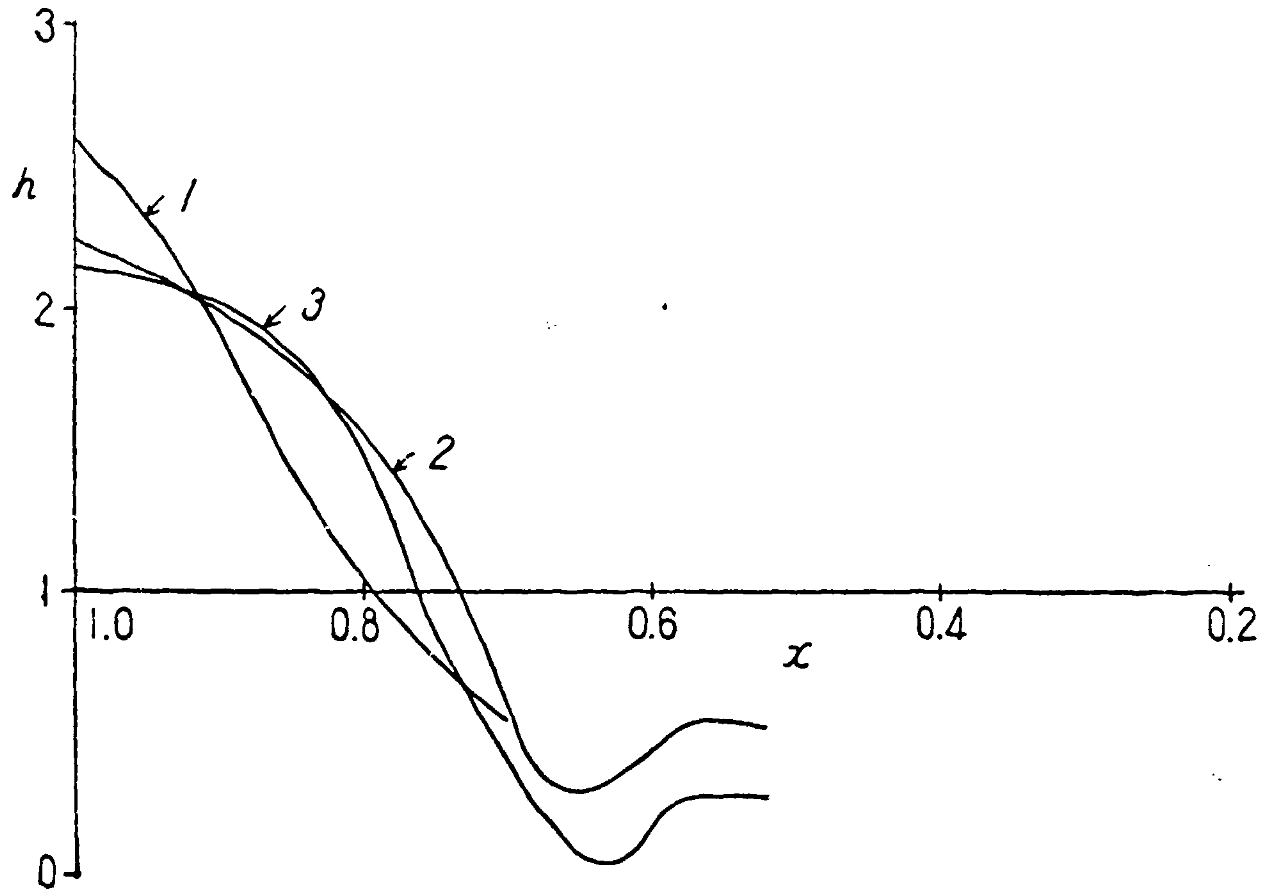

Figure 13
Experimental h vs x distributions for case II-4 (Ōshima, 1960)

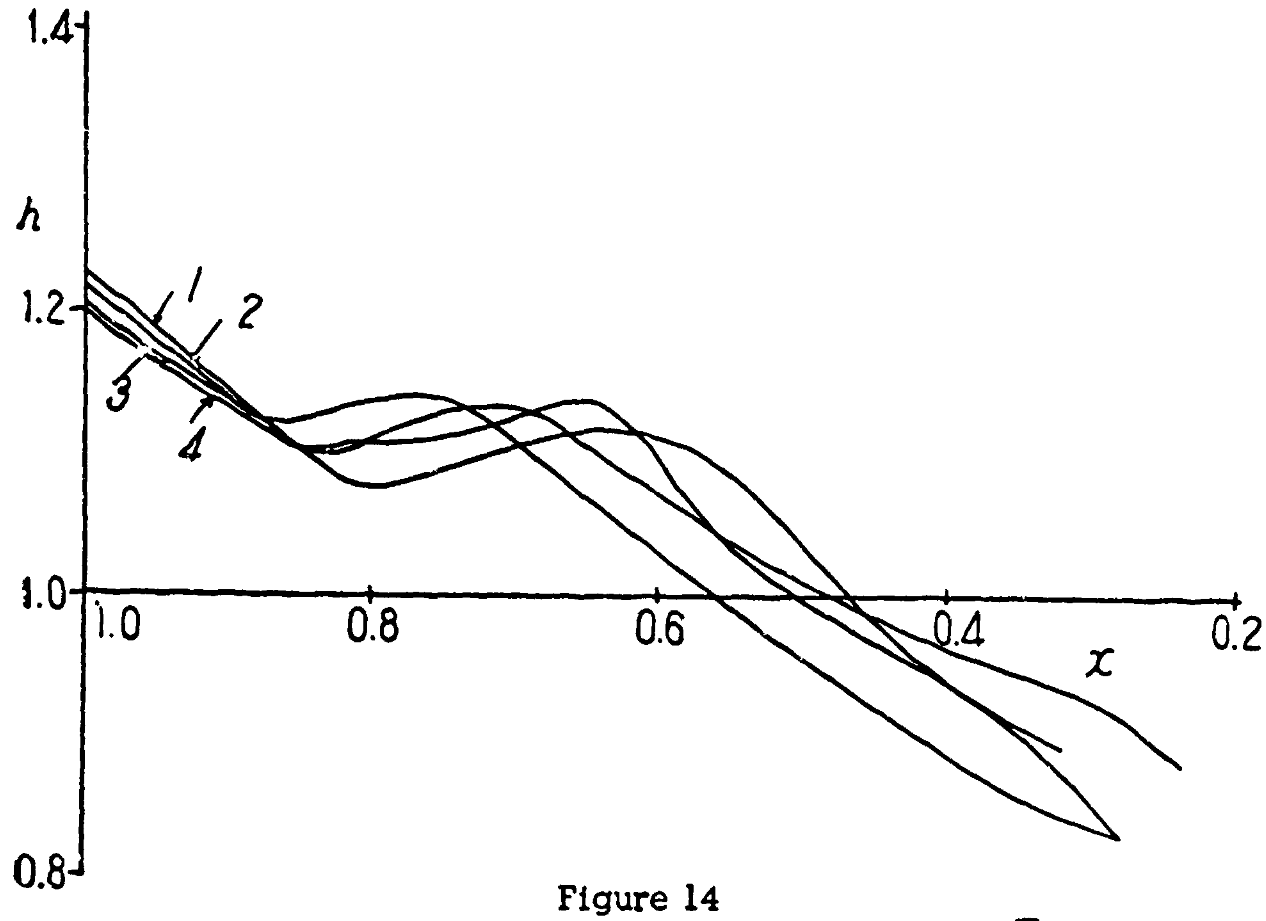

Figure 14
Experimental h vs x distributions for case IV-1 (Ōshima, 1960)

in the region of x . At the next stage, with Mach numbers from about 2 down to about 1.15 (cases II, III) (See Figure 13), the rarefaction type region occupies a considerable part of the center of the blast wave. However, the flow field of the remaining part agrees well with the point source theory as long as we use solutions valid in the wider range of R (or y) given in Section 2.4) rather than those given in the strong-shock theory. (Particularily Ōshima used his solution given in Section 2.4 for the purpose). At the later stage of the weak shock region such as in cases IV, VIII (the Mach number is lower than 1.15 as shown in Figure 14) the flow pattern is very much complicated by the effects of the succeeding rarefaction waves and the secondary shock waves.

Good agreements between the theoretical h-curve and the observed density distribution in the range of Mach numbers above, suggest that we may determine the energy value by utilizing the theoretical relation and the experimental data of the distance R and the arrival time t . Ōshima did this by plotting the data in a diagram of $R - R/R_0$ in the following way: First, $U = dR/dt$ values are computed from the $R - t$ data so that we get an experimental relation between R and U/c . Secondly, theoretical R/R_0 values can be found for each U/c values from the theoretical $U - R$ relation as given in Eq. (2.22) or (Eq. (2-57), Eq. (2-68)). Suppose we plot these experimental R values and theoretical R/R_0 values corresponding to the same U/c values. We may expect a linearty in the (R) exp. - (R/R_0) theo. diagram as far as the phenomenon carries the characteristics predicted by the point (line in this case) explosion theory. The diagram given by Ōshima, using the second approximation formula Eq. (2-30) as the theoretical U-R relation, is reproduced in

Figure 15. In this diagram we can see the linearity especially for the cases of stronger shocks. It is noted that the deviation from the linearity in the figure

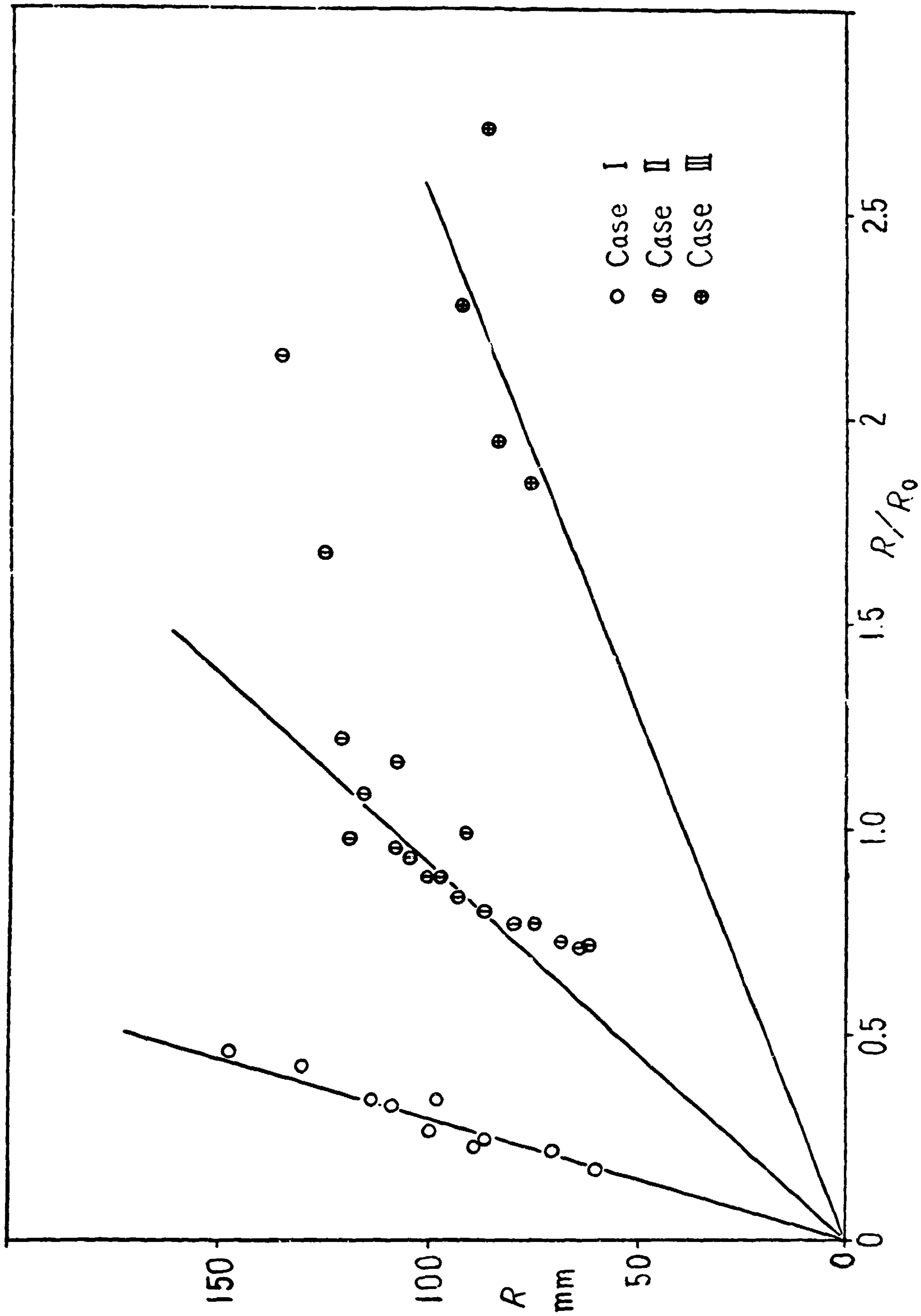

Figure 15

Relations between r/R_0 and R based on the second approximation in Eq. (2.30) (Ōshima, 1960)

does not necessarily mean the failure of the point source theory but instead it reveals the inadequacy of the second approximation to this region. The slopes of these lines give the values of R_0 for respective cases. From the values of R_0, the explosion energy (or, more precisely, the energy equivalent to produce the blast wave generated from a line source) can be obtained from the formula $E = 2\pi p_0 R_0^2$. To check the E values thus obtained, Ōshima estimated the E values by use of a more direct method of evaluating the energy integral given in Eq. (2-8). This method used the observed density distribution ρ and the values of u, p, estimated from the ρ values. These E values agree very well with those obtained above.

3.4 Magnetohydrodynamics

Essential features of the blast wave phenomena in magnetohydrodynamics are expected to be similar in nature to those discussed above for non-conducting fluids as both resulted from the same sort of non-linearity. However some modifications are needed to represent the interaction between velocity and magnetic fields. In general two different approaches are used depending on whether or not electrical conductivity is involved. In the case of low conductivity, which is usually the situation in laboratory experiments, the modification of the theory is more or less straightforward and the perturbation method may effectively be used. It should be noted, however, that the procedure is not uniformally valid, but is valid only in the finite region of the space. This procedure will be given in the subsection 3.4.1, where the discussion will be confined to the specified type of flow preserving axial symmetry. This is rather typical in this field of magnetohydrodynamics, since this is conveniently realized with

axial or azimuthal magnetic fields in connecting with recent experiments on pinch effects, exploding wire, etc.

The case of high conductivity may well be approximated by the limiting case of infinite electrical conductivity, more exactly, infinite magnetic Reynolds number. Since the magnetic field is "frozen" in the fluid in this case, the assumption simplifies the situation. However, this sometimes brings about confusion leading to erroneous conclusions and hence we must proceed with caution along this line. It is possible to have a completely different situation depending on whether or not the initial magnetic field is permeated in the fluid. These will be the subject of the Subsection 3.4.2.

3.4.1 Blast Wave Phenomena in the Fluids of Low Conductivity

We consider here the cylindrical type flow. The magnetic and electric fields $\mathbf{B}$, $\mathbf{E}$ consistent with and preserving the symmetry are given by

$$\mathbf{B} = (0,\ B_\theta,\ B_z),\quad \mathbf{E} = (0,\ E_\theta,\ E_z)$$

where B_θ, E_θ; B_z, E_z are the azimuthal and the axial components and are assumed functions of r and t only. The equations of motion, continuity and energy given in Eqs. (2.1), (2.2), (2.4) are now modified to give

$$\rho\frac{Du}{Dt} = -\frac{\partial p}{\partial r} - \frac{B_z}{\mu}\frac{\partial B_z}{\partial r} - \frac{B_\theta}{\mu r}\frac{\partial}{\partial r}(r B_\theta)\ , \tag{3.18}$$

$$\frac{D\rho}{Dt} = -\rho\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right)\ , \tag{3.19}$$

$$\frac{Dp}{Dt} = -\gamma p\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right) + \frac{\gamma-1}{\mu^2\sigma}\left\{\left(\frac{1}{r}\frac{\partial}{\partial r}r B_\theta\right)^2 + \left(\frac{\partial B_z}{\partial r}\right)^2\right\}\ , \tag{3.20}$$

where σ is the electrical conductivity, μ is the magnetic permeability and they are assumed constant. These are supplemented by the Maxwell equations,

$$\frac{\partial E}{\partial r} = \frac{\partial B_\theta}{\partial t} \quad , \tag{3.21}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rE_\theta) = -\frac{\partial B_z}{\partial t} \quad , \tag{3.22}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rB_\theta) = \mu\,\sigma(E_z + u\,B_\theta) \quad , \tag{3.23}$$

$$\frac{\partial B_z}{\partial r} = \mu\,\sigma(u\,B_z - E_\theta) \quad . \tag{3.24}$$

Since we assume σ is small, the ordinary shock condition (2.6) may be used at the front $r = R$, and the magnetic and electric fields $\mathbf{B}$, $\mathbf{E}$ are continuous there, thus

$$(\mathbf{B})_{r=R} = \mathbf{B}_0, \quad (\mathbf{E})_{r=R} = \mathbf{E}_0 \tag{3.25}$$

where $\mathbf{B}_0$, $\mathbf{E}_0$ are the values of $\mathbf{B}$, $\mathbf{E}$ in front of the shock wave presumably known. They may be functions of R.

Suppose the magnitudes of the given field $\mathbf{B}_0$, $\mathbf{E}_0$ are small or of the ordinary order in a sense such that its magnetic pressure is smaller than or at least comparable with the pressure P_0, then it may then be expected that the interaction effects of these fields to the flow are small. In fact Eqs. (3.23), (3.24) show that the qualities

$$\frac{1}{r}\frac{\partial}{\partial r}(r\,B_\theta), \; \frac{\partial B_z}{\partial r}$$

are small when σ is small and deviations in B_θ, B_z from their unperturbed values $B_{\theta 0}$,B_{z0} are thus expected to be small. The first approximation to these deviations may be found by replacing those qualities $\mathbf{B}$, $\mathbf{E}$, u, (involved in terms multiplied by σ) with their unperturbed values $\mathbf{B}_0$, $\mathbf{E}_0$ and u (given by the solution of corresponding non-conducting motion). Thus we get from Eqs. (3.18), (3.20), (3.23), (3.24),

$$
\begin{cases}
\rho \dfrac{Du}{Dt} = -\dfrac{\partial p}{\partial r} - \sigma \left[B_{z0}(uB_{z0} - E_{\theta 0}) + B_{\theta 0}(E_{z0} + uB_{\theta 0}) \right] , \\[2ex]
\dfrac{D}{Dt} p\rho^{-\gamma} = \sigma(\gamma - 1)\ \rho^{-\gamma} \left[(E_{z0} + uB_{\theta 0})^2 + (uB_{z0} - E_{\theta 0})^2 \right] , \\[2ex]
\dfrac{1}{r} \dfrac{\partial}{\partial r}(rB_\theta) = \mu\sigma (E_{z0} + uB_{\theta 0}) , \\[2ex]
\dfrac{\partial B_z}{\partial r} = \mu\sigma (uB_{z0} - E_{\theta 0}) .
\end{cases}
\tag{3.26}
$$

The last two equations of Eqs. (3.26) give the disturbance in the magnetic field due to the interaction and thus we have

$$
\begin{cases}
rB_\theta = \mu\sigma \displaystyle\int_R^r (E_{z0} + uB_{\theta 0})\, rdr + RB_{\theta 0} , \\[2ex]
B_z = \mu\sigma \displaystyle\int_R^r (uB_{z0} - E_{\theta 0})\, dr + B_{z0} .
\end{cases}
\tag{3.27}
$$

The rest of the equations in Eq. (3.26), supplemented by Eq. (3.19), provide a system to determine u, p, ρ . The system may be solved in a fashion similar to Section 2.3 by expanding the solution in powers of y . After which the deviation due to the magnetic field comes to the order of σ in their respective terms as $f^{(i)}$, $g^{(i)}$, $h^{(i)}$ $(i = 0, 1, \ldots)$.

This method has been used to estimate the effects of the applied axial magnetic field to the cylindrical blast wave produced by exploding a fine wire (Sakurai, 1962a, Sakurai and Takao, 1963, 1964), in which $\mathbf{B}_0$, $\mathbf{E}_0$ are assumed as

$$
B_{z0} = B_0, \quad B_{\theta 0} = \mathbf{E}_0 = 0
$$

and B_z given in Eq. (3.27) is simplified to,

$$
B_z = B_0 \left(1 + \mu\sigma \int_R^r u\, dr\right) . \tag{3.28}
$$

Suppose we assume the unperturbed flow u is given blast wave theory in Eq. (2.11). Then Eq. (3.28) has the more explicit form

$$B_z = \begin{cases} B_0 (1 + \mu\sigma \ UR \int_1^x f(x,y)\,dx) & , \ x \leq 1 \\ B_0 & , \ x > 1 , \end{cases} \tag{3.29}$$

where the expressions (2.10) and (2.11) have been used. Utilizing the series expansion solution (Section 2.3) we have

$$f = f^{(0)} + y\, f^{(1)} + \ldots ,$$

$$\left(\frac{C\, R_0}{U R}\right)^2 = J_0 \, [1 + \lambda_1 y + \ldots] ,$$

and Eq. (3.2) gives

$$\frac{B_z}{B_0} = 1 + R_m \, [\int_1^x f^{(0)} dx + y(\int_1^x f^{(1)} dx - \frac{1}{2} \lambda_1 \int_1^x f^{(0)} dx) + \ldots] , \quad x \leq 1 ,$$

where we have defined the magnetic Reynolds number R_m as

$$R_m = \frac{\sigma\, \mu c R_0}{\sqrt{J_0}} .$$

The values of B_z/B_0 computed to the first term (neglecting $0(y)$ terms) are plotted in Figure 16. Electric field E_θ, derived from Eq. (3.23) using the expression for B_z above becomes singular at $x = 0$ as $0(1/x)$. This shows the procedure is not uniformly valid, as noticed above. Perturbation equations for u, p, given in Eq. (3.26) become in this case,

$$\rho \frac{Du}{Dt} = - \frac{\partial p}{\partial r} - \sigma \ B_0^2 u ,$$

$$\frac{Dp}{Dt} = - \gamma p (\frac{\partial u}{\partial r} + \frac{u}{r}) + \sigma (\gamma - B_0^2 \, u^2) , \tag{3.30}$$

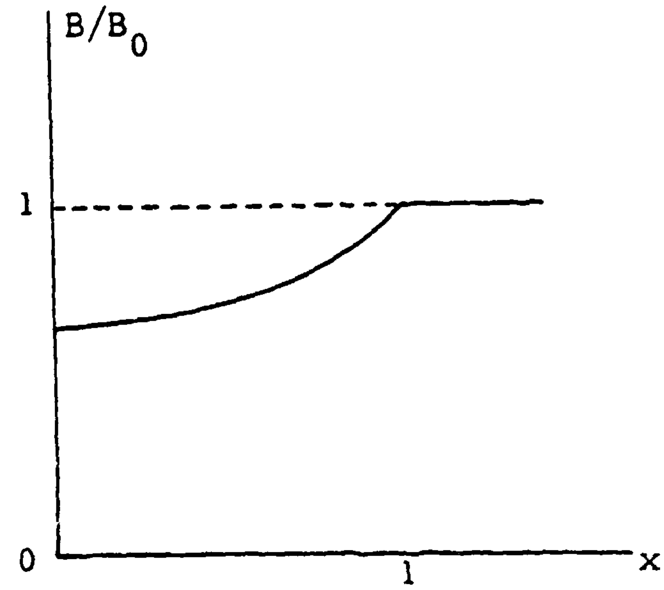

Figure 16

Decrease in axial magnetic field due to its interaction with cylindrical blast wave

where u (in the terms multiplied by σ) is supposed to be given by

$$u = U\, f(x,y) = U(f^{(0)} + y\, f^{(1)} + \dots) \quad .$$

It is convenient to introduce new dependent functions $\bar{f}$, $\bar{g}$, $\bar{h}$ defined by

$$u = U\bar{f}(x,y), \quad p = p_0\, y^{-1}\, \bar{g}(x,y), \quad \rho = \rho_0\, \bar{h}(x,y) \quad ,$$

where upon Equations (3.30) are transformed into,

$$\begin{cases} \bar{h}\{(\bar{f}-x)\dfrac{\partial \bar{f}}{\partial x} + \lambda y \dfrac{\partial \bar{f}}{\partial y} - \dfrac{1}{2}\lambda \bar{f}\} = -\dfrac{1}{\gamma}\dfrac{\partial \bar{g}}{\partial x} - \sigma\mu\, UR\, \dfrac{B_0^2}{\gamma\mu p_0}\, y\, f \quad , \\[2ex] -\lambda\, \bar{g} + (f-x)\dfrac{\partial \bar{g}}{\partial x} + \lambda y \dfrac{\partial \bar{g}}{\partial y} = -\gamma \bar{g}\,(\dfrac{\partial \bar{f}}{\partial x} + \dfrac{\bar{f}}{x}) \\[2ex] \qquad\qquad + (\gamma - 1)\,\sigma\mu\, UR\, \dfrac{B_0^2}{\mu p_0}\, y\, f^2 \quad . \end{cases} \tag{3.31}$$

By expanding $\bar{f}$, $\bar{g}$, $\bar{h}$, in a power series in y ,

$$\bar{f} = \bar{f}^{(0)} + y\bar{f}^{(1)} + \dots, \quad \bar{g} = \bar{g}^{(0)} + y\,\bar{g}^{(1)} + \dots, \quad \bar{h} = \bar{h}^{(0)} + y\,\bar{h}^{(1)} + \dots \quad ,$$

and utilizing the relation

$$\sigma\mu\, UR = R_m\,[1-\frac{1}{2}\lambda_1 y + \dots] \ ,$$

we get systems of equations from which to determine $f^{(i)}$, $g^{(i)}$, $h^{(i)}$, $i=0,1,2,\dots$. It is observed that the terms multiplied by σ in Eq. (3.31) all originated from the first power of y in their power series expansion. Therefore the deviations in $\bar{f}$, $\bar{g}$, $\bar{h}$ from the unperturbed f, g, h do not appear in the first approximation which neglects terms of the order of y (strong shock approximation). The same kind of perturbation process may be applied to other cases such as

$$B_{\theta 0} = \frac{S}{r}, \quad B_{z0} = E_0 = 0 \ , \tag{3.32}$$

which represents a situation associated with the problem of a blast wave influenced by a magnetic field produced by an electric current flowing through the center axis. It is not necessary that the current be constant with time only that it is a slowly varying function. More complicated situation like

$$B_{z0} = B_0, \ B_{\theta 0} \propto 1/r, \ \ E_0 = 0 \ ,$$

may also be treated in the same manner. Note that this case is important to the study on the effects of the applied axial magnetic field to the exploding wire phenomena, and differs from the previous case as $B_{\theta 0}$ is not zero. This component should be considered in the study of the early stage of the phenomenon, since the electric current of the exploding wire produces azimuthal magnetic field for a while.

It is noted that the case given by Eq. (3.32) can be represented by a similarity solution under the assumption of strong shock and s = constant (Greenspan, 1962, see also Korobeinikov and Ryazanov, 1962). The solution has the following form,

$$u = U f(x)\,,\ p = p_0 \frac{U^2}{C^2} g(x)\,,\ \rho = \rho_0 h(x)\,,\ B_\theta = \frac{\mu I}{2\pi} \frac{b(x)}{R}\ . \tag{3.33}$$

While the expressions for u, p, ρ above are the same as in the ordinary blast wave theory, the expression for B_θ is obtained so as to fit the boundary condition to B_θ in Eq. (3.32), which can be expressed as

$$B_{\theta 0} = \frac{\mu I}{2\pi R}\ , \qquad \text{I: Electric current}$$

and accordingly we have,

$$b(1) = 1\ .$$

Substituting Eqs. (3.33) into Eqs. (3.18), (3.19) and (3.20), we get equations similar to Eqs. (2.25):

$$\begin{cases} h\left(-\frac{\lambda}{2} f + (f-x)\frac{df}{dx}\right) = -\frac{1}{\gamma}\frac{dg}{dx} - \delta \cdot \frac{b}{x}\frac{d}{dx}(xb)\ , \\ (f-x)\frac{dh}{dx} = -h\left(\frac{df}{dx} + \frac{f}{x}\right)\ , \\ -\lambda g + (f-x)\frac{dg}{dx} + \gamma g\left(\frac{df}{dx} + \frac{f}{x}\right) = \gamma(\gamma-1)\,\delta \overline{R_m}\left[\frac{1}{x}\frac{d}{dx}(xb)\right]^2\ , \end{cases} \tag{3.34}$$

where we have defined

$$\delta = \frac{\mu I^2}{4\pi^2 \rho_0 R^2 U^2}, \quad \overline{R_m} = \frac{1}{\sigma \mu R U}\ . \tag{3.35}$$

Eliminating E_z from Eqs. (3.21) and (3.23), we get an equation for b

$$R_m \frac{d}{dx}\left[\frac{1}{x}\frac{d}{dx}(xb)\right] = \frac{d}{dx}\left[(f-x)b\right]\ . \tag{3.36}$$

It is readily seen that these equations (3.34) and (3.36) are consistent with the boundary conditions (2.27) based on the strong shock assumption if RU = const. (= A, say), from which we get

$$\lambda = \frac{R}{y}\frac{dy}{dR} = 2\,,\quad \delta,\ \overline{R_m} = \text{const.}$$

It turns out that the system of equations have two intermediate integrals; one follows directly from Eq. (3.36) and the other one from the energy consideration. These are

$$\overline{R}_m \frac{1}{x}\frac{d}{dx}(xb) = (f-x)b + K \ ,$$

$$-x^2 \left[\frac{q}{\gamma(\gamma-1)} + \frac{1}{2} h f^2\right] + xf\left[\frac{q}{\gamma-1} + \frac{1}{2} h f^2\right] - \delta \overline{R}_m \ b \frac{d}{dx}(bx) = K' \ ,$$

where K, K' are the integration constants. For further developments in seeking the solution of the system, it was found that the four conditions given in Eqs. (2.27) and $b(1) = 1$ are not enough. An additional condition $[xb(x)]_{x=0} = 1$ (a concentrated unit axial current) must be satisfied to have a similarity solution which is finite and which applies to the entire domain $0 \le x \le 1$.

It is interesting to utilize the solution to check the validity of the approximate solution given by the method considered above for small σ . It was found that both agree well for small σ , except for the region near $x = 0$, where the purterbation for small σ was found to be not valid.

3.4.2 Magnetohydrodynamic Blast Waves

Electrical conductivity is assumed infinite in this section. This limiting case simplifies the problem to approximate the situation of high conductivity. The interaction between the magnetic and the flow fields is strongest in this case and the magnetic field is "frozen" in the fluid. In consequence of this fact, the fluid stays without magnetic field all the time if no magnetic field permeated initially, and the fluid behaves as ordinary non-conducting fluid. The effect of the outside magnetic field comes only through the bounding surface acting as magnetic pressure. An example of this type of flow associated with the blast

wave theory is seen in the analysis on the transient state of plasma pinch given by Kuwabara (1958, 1963). In this example the plasma is assumed to be a perfect conductor and there is no magnetic field permeated initially in the plasma column. Two different configurations of self-and induced-pinch are considered simultaneously. These are caused either by an azimuthal or axial magnetic field, but they act only as magnetic pressure to press the plasma column inwards at $r = R_2$ (c.f. Figure 17). This results in a cylindrical shock wave at $r = R_1$, which

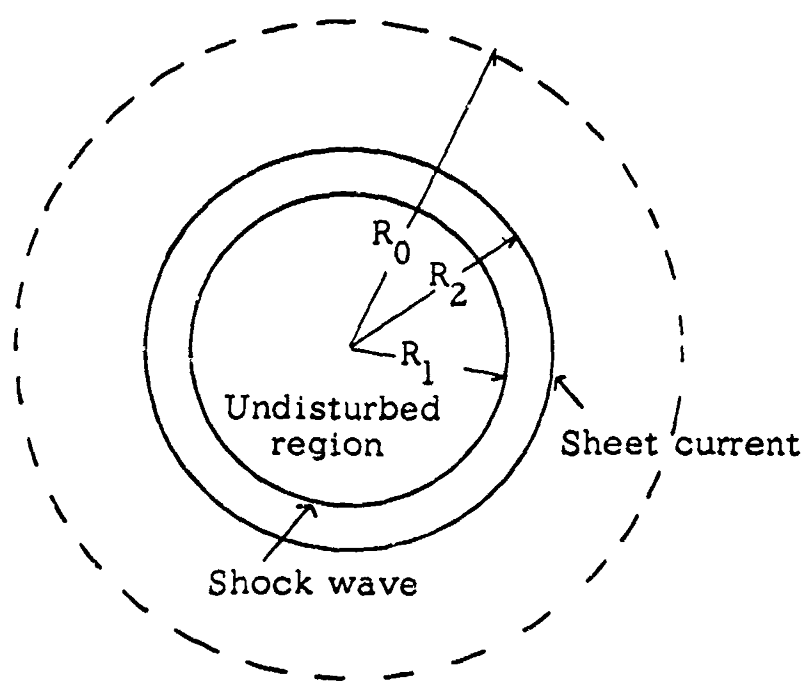

Figure 17

Cross section of the plasma cylinder

converges to the center axis. The motion of the plasma is purely hydrodynamic as described by Eqs. (2.1), (2.2) and (2.3). Assuming the strong shock condition (2.27) and the similarity solution given by Eq. (2.25) (in the range of $x \geq 1$, different from the blast wave case of $0 \leq x \leq 1$), Kuwabara examined the

sheet current at $r = R_2$ expected from the similarity configuration. He also examined the converging feature of the column, which revealed that the amount $(R_2-R_1)/R_2$ stays small during the process of pinch. This fact is consistent with the assumption considered by Allen (1957) in which he analized the problem by use of the snow plough model.

With magnetic field permeated initially, the fundamental equations for the cylindrical type flow given in Eqs. (3.18)-(3.24) are simplified under the assumption of infinite conductivity and we get from Eqs. (3.20)-(3.24),

$$\begin{cases} \dfrac{D}{Dt} p\rho^{-\gamma} = 0 \ , \\ \dfrac{\partial B_\theta}{\partial t} + \dfrac{\partial}{\partial r} u B_\theta = 0, \quad E_z + u B_\theta = 0 \ , \\ \dfrac{\partial B_z}{\partial t} + \dfrac{1}{r} \dfrac{\partial}{\partial r}(ru B_z) = 0, \quad E_\theta - u B_z = 0 \ , \end{cases} \tag{3.37}$$

while Eqs. (3.18) and (3.19) are unchanged.

Two different cases must be distinguished depending on the condition at the shock front. First, we consider the case of $\sigma = \infty$ inside the blast wave but $\sigma = 0$ outside, which may correspond to the case of ionizing shock front. Since the magnetic and the electric field should remain continuous in this case, we have the ordinally shock condition (2.6) to give the jumps in the velocity, the pressure and the density. The second case assumes $\sigma = \infty$ everywhere, where upon the magnetic field may be discontinuous at the shock front resulting from a sheet current there. More details of the shock condition associated with this case may be found from the following conservation relations of mass, momentum and energy,

$$\begin{cases}[\rho(U-u)]_{r=R} = \rho_0 U , \\ [p + \rho(U-u)^2 + \frac{1}{2\mu} B^2]_{r=R} = p_0^2 + \rho_0 U^2 + \frac{1}{2\mu} B_0^2 , \\ [\rho(u-U)\{\frac{1}{2}(u-U)^2 + \frac{\gamma}{\gamma-1}\frac{p}{\rho}\} + \frac{1}{\mu} B^2(u-U)]_{r=R} \\ \qquad = -\rho_0 U(\frac{1}{2}U^2 + \frac{\gamma}{\gamma-1}\frac{p_0}{\rho_0}) - \frac{1}{\mu} B_0^2 U ,\end{cases}$$

supplemented by the "frozen" condition,

$$[\rho/B]_{r=R} = \rho_0/B_0 .$$

The conditions above lead to the following conditions

$$(u)_{r=R}/U = 1 - \frac{1}{2}\frac{\gamma-1}{\gamma+1}\{1 + \frac{2}{\gamma-1} y^* + \sqrt{(1 + \frac{2}{\gamma-1} y^*)^2 + \frac{4(\gamma+1)(2-\gamma)}{(\gamma-1)^2} A^{-2}}\} ,$$

$$\begin{cases}(p^*)_{r=R}/p_0^* = 1 + \gamma[u]_{r=R}/(U y^*) , \\ \rho_0/(\rho)_{r=R} = B_0/(B)_{r=R} = 1 - [u]_{r=R}/U ,\end{cases} \tag{3.38}$$

where we have defined

$$p^* = p + \frac{1}{2\mu} B^2, \quad y^* = C^{*2}/U^2, \quad C^{*2} = \gamma p^*/\rho, \quad A = U\sqrt{\rho_0\mu}/B_0 .$$

It is interesting to notice that the condition (3.38) in the special case of $\gamma = 2$, we have

$$\begin{cases}(u)_{r=R} = \frac{2}{3} U(1 + y^*) , \\ (p^*)_{r=R} = \frac{1}{3} p_0^*(4 y^{*-1} - 1) , \\ (\rho)_{r=R} = 3\rho_0(1 + 2y^*)^{-1} ,\end{cases} \tag{3.39}$$

which has the same form as the ordinary shock condition given in Eq. (2.6) for $\gamma = 2$, provided p is replaced by p^*. It should be recalled that $\gamma = 2$ is

known to approximate the plasma state in a uniform magnetic field (c. f. Section 2. 3. 2).

Among the various specialized configurations associated with applications, we first consider the case,

$$B_{z0} = \text{const.}, \quad B_{\theta 0} = 0, \quad E_0 = 0$$

which corresponds to the situation above in the problem of estimating the effects of applied axial magnetic field on the blast wave from exploding wire (Sakurai, 1962b). σ is assumed infinite everywhere and the shock condition is given by Eq. (3. 38). Equations (3.18), (3.19) and (3. 20) are simplified in this present configuration to give,

$$\begin{cases} \dfrac{Du}{Dt} = -\dfrac{1}{\rho}\dfrac{\partial}{\partial r}\left(p + \dfrac{1}{2\mu}B_z^2\right) , \\ \dfrac{1}{\rho}\dfrac{D\rho}{Dt} = \dfrac{1}{\gamma p}\dfrac{Dp}{Dt} = \dfrac{1}{B_z}\dfrac{DB_z}{Dt} = -\dfrac{1}{r}\dfrac{\partial}{\partial r}(ur) . \end{cases} \tag{3. 40}$$

The problem is simplified with a further assumption of $\gamma = 2$ and Eqs. (3. 40) become

$$\frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial p^*}{\partial t}, \quad \frac{1}{\rho}\frac{D\rho}{Dt} = \frac{1}{2p^*}\frac{Dp^*}{Dt} = -\frac{1}{r}\frac{\partial}{\partial r}(ur) . \tag{3. 41}$$

The system of equations (3. 41) is the same as that given by Eqs. (2. 1), (2. 2) and (2. 3) for ordinary non-conducting fluid with $\alpha = 1$, $\gamma = 2$. Thus this system supplemented by the condition (3. 39) provides the same solution as considered in Chapter 2 on the blast wave. So we can get the cylindrical blast wave solution under the axial magnetic field by simply replacing p by p^* in the solution given in Chapter 2. It turns out that Eq. (3. 41) with Eq. (3. 39) represents a more general motion of the cylindrical shock wave of a progressive type (Courant and Friedrichs, 1948) and all solutions given for the ordinary gas are valid in this

case. In fact Kuwabara (1962) used this system to analyze the transient state of pinch phenomenon of plasma column in which the axial magnetic field is permeated initially. Many other problems may be treated by this scheme by choosing $\gamma = 2$ to get a rough idea of the features.

It is also known in this case of $\gamma = 2$ that the more general situation of

$$B_{z0} = \text{const.}, \quad B_{\theta 0} \propto 1/4, \quad E_0 = 0 \quad ,$$

can be represented by the similarity solution for the strong shock assumption (Korobeinikov, 1962). Also the similarity solution for general values of γ is possible if the magnetic field is purely azimuthal such as given by

$$B_{z0} = 0, \quad B_{\theta 0} \propto 1/r, \quad E_0 = 0 \quad ,$$

(Pai, 1958, Cole and Greifinger, 1962, Korobeinikov, 1962).

There is another interesting application of blast wave theory to the phenomenon called "inverse pinch" (Anderson et. al. 1958, Korobeinikov and Ryazanov, 1960, Cole and Greifinger, 1961, Liepmann and Vlases, 1962). In this situation we have an applied axial magnetic field permeated in an ideal gas of infin... conductivity, and an electric current through the axis (wire does not explode). Thus we get an expanding cylindrical vacuum region pushed by the magnetic pressure produced from an axial electric current on the cylindrical surface which is returning along the axis. The magnetic pressure acts as a cylindrical piston and results an expanding cylindrical shock wave ahead. It is interesting to note that there enters no characteristic length or time in the initial or boundary conditions of this problem. Thus the solution should have the property of psuedo-steady. In which case it could be expressed by a similarity solution of blast wave type, where the shock velocity is constant and the electric current is proportional to the time.

3.5 Cavitation

By way of studing the cavitation problem in water, Hunter (1960) introduced a similarity solution in the final stage of a collapsing empty spherical cavity. The similarity however requires that the velocity of sound vanishes at the cavity wall. Although this requirement is satisfied with the fluids obeying a polytropic law of the form $p \propto \rho^m$ with m the index (Hunter, 1963), it is usually not the case in water where the equation of state is conveniently given by the Tait-equation,

$$\frac{p+B}{B} = \left(\frac{\rho}{\rho_0}\right)^\gamma ,$$

where B is a slowly varying function of entropy and γ has a value of 7, say. The velocity of sound c is given by

$$c^2 = \left(\frac{dp}{d\rho}\right)_{\text{entropy}} = \frac{\gamma B \rho^{\gamma-1}}{\rho_0^\gamma} ,$$

and it does not vanish at the cavity wall since ρ remains finite as the pressure p goes to zero there. So the similarity solution satisfies the boundary condition only in the limiting case of $p \gg B$ and when the radius of the bubble R becomes small (or dR/dt becomes large). To improve the solution for larger R, and still take into account the effect of the finite density at the cavity wall, Holt and Schwarz (1963) used a perturbation method. This method involved expanding the solution in a series of $\left(\frac{dR}{dt}\right)^{-1}$ in a fashion similar to that given in Section 2.3. It turns out that the procedure is not straightf[illegible] because of singularities involved, and must be modified by introducing Lighthill's technique (1949) to avoid the singularities.

The velocity and the velocity of sound are given by the following equations of momentum and continuity,

$$\begin{cases} \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial r} + \dfrac{1}{\gamma-1}\dfrac{\partial c^2}{\partial r} = 0 \ , \\ \dfrac{\partial c^2}{\partial t} + u\dfrac{\partial c^2}{\partial r} + (\gamma-1)c^2\left(\dfrac{\partial u}{\partial r} + \dfrac{2u}{r}\right) = 0 \ , \end{cases} \tag{3.42}$$

where u, c^2, r, t have been expressed in dimensionless forms. Equations (3.42) are supplemented by the boundary conditions,

$$u = \dot{R}\,, \quad c = 1 \text{ at } r = R \ . \tag{3.43}$$

It was shown by Hunter (1960) that Eqs. (3.42) have a similarity solution such that $u/\dot{R}$, $c^2/\dot{R}^2$ are functions of r/R ($=x$) only, and $R\ddot{R}/\dot{R}^2 = K$, (3.44)

where K is a constant to be determined from the regularity of the solution. Now the solution $u/\dot{R}$ satisfies the boundary condition $u = \dot{R}$ by setting $(u/\dot{R})_{x=1} = 1$, while $(c^2/\dot{R}^2)_{x=1}$ has to be put zero since $\dot{R}\to\infty$ as $R\to 0$, which is however compatible with the condition $c = 1$ only for small R. To find the solution valid for a wider range of R, Holt and Schwartz (1963) introduced a transformation of variables similar to that given in Section 2.2,

$$\begin{aligned} &u = \dot{R}_H f(x,y)\,, \quad c^2 = \dot{R}_H^2 g(x,y) \ , \\ &x = \frac{r}{R_H}\,, \quad y = \frac{(c)_{r=R}}{\dot{R}_H} = \frac{1}{\dot{R}_H} \ , \end{aligned} \tag{3.45}$$

where R_H signifies Hunter's R defined in Eq. (3.44). Equations (3.43) are then transformed into,

$$\begin{cases} Kf + (f-x)\dfrac{\partial f}{\partial y} - Ky\dfrac{\partial f}{\partial y} = -\dfrac{1}{\gamma-1}\dfrac{\partial g}{\partial x} \ , \\ 2Kg + (f-x)\dfrac{\partial g}{\partial y} - Ky\dfrac{\partial g}{\partial y} = -(\gamma-1)\,g\left(\dfrac{\partial f}{\partial x} + \dfrac{2f}{x}\right) \ , \end{cases} \tag{3.46}$$

which are very similar to Eqs. (2.15) and their solution for small y may be found in a similar manner as in Section 2.3.

Expanding f and g in series of y^2,

$$\begin{cases} f = f^{(0)}(x) + y^2 f^{(1)}(x) + y^4 f^{(2)}(x) + \dots , \\ g = g^{(0)}(x) + y^2 g^{(1)}(x) + y^4 g^{(2)}(x) + \dots , \end{cases} \qquad (3.47)$$

we get from Eqs. (3.46) the following systems of equations,

$$\begin{cases} [(f^{(0)}-x)^2 - g^{(0)}]\dfrac{df^{(0)}}{dx} = -Kf^{(0)}(f^{(0)}-x) + \dfrac{2}{\gamma-1}g^{(0)}(K+\dfrac{\gamma-1}{x}f^{(0)}) , \\ [(f^{(0)}-x)^2 - g^{(0)}]\dfrac{dg^{(0)}}{dx} = (\gamma-1)Kf^{(0)}g^{(0)} - 2g^{(0)}(f^{(0)}-x)(K+\dfrac{\gamma-1}{x}f^{(0)}) , \end{cases} \qquad (3.48)$$

$$\begin{cases} [(f^{(0)}-x)^2 - g^{(0)}]\dfrac{df^{(1)}}{dx} = \beta_1(x)\, g^{(1)} + \beta_2(x)\, f^{(1)} , \\ [(f^{(0)}-x)^2 - g^{(0)}]\dfrac{dg^{(1)}}{dx} = \beta_3(x)\, g^{(1)} + \beta_4(x)\, f^{(1)} , \\ \dots , \end{cases} \qquad (3.49)$$

where $\beta_1(x), \dots, \beta_4(x)$ are functions given by $f^{(0)}$, $g^{(0)}$. R is considered as a function of y only and is expanded as

$$R = R_H(1 + \alpha_1 y^2 + \dots) , \qquad (3.50)$$

where $\alpha_1, \dots$ are constants to be determined.

Utilizing the expansion formulae given in Eqs. (3.50) and (3.47), the boundary condition in Eq. (3.43) are expanded as,

$$u = \dot{R}_H[f^{(0)}(1) + y^2\{f^{(0)\prime}(1)\alpha_1 + f^{(1)}(1)\} + \dots] = \dot{R}_H[1 + y^2\alpha_1(1-2K) + \dots] ,$$

$$(1/y^2)[g^{(0)}(1) + y^2\{g^{(0)\prime}(1)\alpha_1 + g^{(1)}(1)\} + \dots] = 1 ,$$

from which we get

$$f^{(0)}(1) = 1, \quad g^{(0)}(1) = 0 ; \qquad (3.51)$$

$$f^{(1)}(1) + \alpha_1 f^{(0)\prime}(1) = \alpha_1(1-2K), \quad g^{(1)}(1) + \alpha_1 g^{(0)\prime}(1) = 1 ; \qquad (3.52)$$

$\dots$.

The first approximation of the problem is given by Eqs. (3.48), (3.51). Unlike the blast wave theory in Section 2.2, Eqs. (3.48) have two singularities resulting from zeros of the denominator $(f^{(0)}-x)^2 - g^{(0)}$, which are one at $x = 1$ and the other one is somewhere at $x = x_s > 1$ (say). A singularity at $x = x_s$ appears also in the blast wave equation (2.25), but we had no difficulty since we were only interested in the region $x \leq 1$. It can be shown that there exists a regular solution at $x = 1$ for any values of K. However a special choice of K makes the solution regular at $x = x_s$, and we can determine the value of K as well as the value of x_s by using this fact. The procedure was performed numerically by Hunter (1960) for $\gamma = 7$ and K, x_s values thus determined are -0.801 and 1.51 respectively. Note that the procedure is similar to that for finding proper solution of some other problems like "converging shock" (Guderley, 1942), "shock wave at the edge of a gas" (Gandel'man and Frank-Kamenetskii, 1956, Sakurai, 1960).

With the value of K as well as the solution $f^{(0)}$, $g^{(0)}$ so determined Eqs. (3.49), (3.52) provide the second approximation to the problem. Note that the factor $(f^{(0)}-x)^2 - g^{(0)}$ appears also in Eqs. (3.49) and becomes zero at $x=1$ and $x=x_s$. The equation is singular there. Since the right hand sides of the equations (3.49) do not vanish at these points. This indicates that the expansion solution in y is not valid there. However, it was found by Holt and Schwartz (1963) that the regular solution near $x = x_s$ could be obtained by modifying the expansion procedure with the use of Lighthill's technique. This involved the changing of the independent variables from x, y to z, y. Where,

$$x = x_s + z + y^2 q_1(z) + y^4 q_2(z) + \dots ,$$

and the functions $q_1, q_2, \ldots$ are to be determined.

The expansion in y is then assumed near $z = 0$ as

$$f = f^{(0)}(z) + f^{(1)}(z)\, y^2 + \ldots$$

$$g = g^{(0)}(z) + g^{(1)}(z)\, y^2 + \ldots .$$

The equations for $f^{(0)}$, $g^{(0)}$ are the same as given in Eq. (3.48) in terms of x. However the equations for $f^{(1)}$, $g^{(1)}$ are different from Eq. (3.49) and are given by

$$\left\{\begin{aligned} &\{[f^{(0)} - (x_s+z)]^2 - g^{(0)}\}\, f^{(1)\prime} \\ &\qquad = \beta_1 g^{(1)} + \beta_2 f^{(1)} + \delta_1 q_1 + \delta_2 q_1' , \\ &\{[f^{(0)} - (x_s+z)]^2 - g^{(0)}\} g^{(1)\prime} \\ &\qquad = \beta_3 g^{(1)} + \beta_4 f^{(1)} + \delta_3 q_1 + \delta_4 q_1' , \end{aligned}\right. \tag{3.53}$$

where $\delta_1, \delta_2, \delta_3, \delta_4$ are all known functions defined by $f^{(0)}$, $g^{(0)}$, K .

We are required to find regular expansion solution of Eq. (3.53) as

$$f^{(1)} = A_0 + A_1 z + \ldots, \quad g^{(1)} = B_0 + B_1 z + \ldots , \tag{3.54}$$

so that the right hand side of the equation (3.53) vanishes at $z = 0$. Substituting the expansion (3.54) into Eq. (3.53) and assuming that q_1=constant so that $q_1' = 0$, we get the following condition,

$$\left\{\begin{aligned} &\bar{\beta}_1 B_0 + \bar{\beta}_2 A_0 + \bar{\delta}_1 q_1 = 0, \\ &\bar{\beta}_3 B_0 + \bar{\beta}_4 A_0 + \bar{\delta}_3 q_1 = 0, \end{aligned}\right. \qquad (\bar{\beta}_i \equiv \beta_i(1),\ \bar{\delta}_i \equiv \delta_i(1),\ i = 1, \ldots 4)$$

from which we have

$$\frac{B_0}{\bar{\beta}_2\bar{\delta}_3 - \bar{\beta}_4\bar{\delta}_1} = \frac{A_0}{\bar{\beta}_3\bar{\delta}_1 - \bar{\beta}_1\bar{\delta}_3} = \frac{q_1}{\bar{\beta}_1\bar{\beta}_4 - \bar{\beta}_2\bar{\beta}_3} , \tag{3.55}$$

where the denominators are all proved to be nonzero at $z = 0$. The higher-order coefficients A_1, B_1, ... can be expressed as known, nonzero multiples of A_0 . Hence, expansions of the form (3.54) can be found in which the coefficients depend on one arbitrary parameter. In this way, one-parameter families of regular integral curves, for $f^{(1)}$ and $g^{(1)}$, can be found in the neighborhood of $z = 0$.

We can also utilize Lighthill's technique to get a regular expansion solution near $x = 1$ and again this solution involves a parameter. The features, however, are slightly different from that for near $x = x_s$ described above.

Assume the expansion near $x = 1$,

$$\begin{cases} x = 1 + \zeta + y^2 q_1(\zeta) + \dots \\ f = f^{(0)}(\zeta) + y^2 f^{(1)}(\zeta) + \dots \\ g = g^{(0)}(\zeta) + y^2 g^{(1)}(\zeta) + \dots \; , \end{cases}$$

which leads to equations for $f^{(1)}$, $g^{(1)}$ similar with Eqs. (3.53). We expect a regular expansion solution near $\zeta = 1$ expressed as

$$\begin{cases} f^{(1)} = C_0 + C_1\zeta + \dots \; , \\ g^{(1)} = D_0 + D_1\zeta + \dots \; , \end{cases} \tag{3.56}$$

and this requires

$$\bar{\beta}_1 D_0 + \bar{\beta}_2 C_0 + \bar{\delta}_1 q_1 = 0 \; , \tag{3.57}$$

$$-g^{(0)}(1)\, D_1 = \bar{\beta}'_3 D_0 + \bar{\beta}'_4 C_0 + \bar{\delta}'_3 q \; . \tag{3.58}$$

We have a relation between C_0 and D_0 given by Eq. (3.52) ,

$$D_0 = 1 + \frac{g^{(0)'}(1)}{2K + f^{(0)'}(1) - 1} C_0 \tag{3.59}$$

which reduces Eq. (3.57) to

$$\frac{1}{\gamma-1}[\gamma f^{(0)'}(1) - 3 + 2(K+\gamma)]\,D_0 - \frac{2K-1+f^{(0)'}(1)}{\gamma-1} + \frac{2K-1}{\gamma-1}C_0 q_1 = 0 \quad . \tag{3.60}$$

It can be shown, however, that

$$\gamma f^{(0)'}(1) - 3 + 2(K+\gamma) = 0$$

and the condition (3.60) is satisfied for any values of D_0 if q_1 is determined by

$$-\frac{2K-1+f^{(0)'}(1)}{\gamma-1} + \frac{2K-1}{\gamma-1}C_0 q_1 = 0 \quad .$$

Since C_0 is determined by Eq. (3.59) for a given D_0 and D_1 is given by Eq. (3.58) and so forth for C_1, D_2, ..., we may have a regular expansion for any value of D_0 .

The parameter D_0 as well as the other parameter A_0 involved in the other solution (3.54) can be determined by fitting the solution Eq. (3.56) with Eq. (3.54) originated from $x = x_s$.

Correction to the cavity wall velocity α_1, which appears in Eq. (3.50) can be obtained from Eq. (3.52), with use of the value of D_0 determined above.

3.6 Thunder

Although one may easily imagine that the thunder is a kind of blast wave generated by lightning, it was realized only very recently that the phenomenon was actually different from the ordinary sound wave in some aspects. The attention to the phenomenon has hitherto been concentrated mostly on the electrical character of the lightning stroke. Also the way of observing this unpredictable occurance of the phenomenon has made it difficult to study the

exact feature of the phenomenon.

Quite recently Kitagawa and Kobayashi (1963) examined their recorded data on the observed thunder and the change in electric field which were both produced by the same lightning. This examination revealed that the time interval between the lightning and the succeeding thunder is in fact shorter than the one expected from the propagation by the sound speed. This fact suggests that thunder is a cylindrical blast wave generated by the lightning as a line source. They also estimated the range in which the propagation velocity was greater than that of sound and concluded that it spreads over a few hundred meters. Encouraged by this fact, they tried to estimate the characteristic length R_0 given in Eq. (2.14). Lightning usually consists of sequential strokes and if we denote the energy from one stroke as W_s, we get from Eq. (2.14),

$$R_0 = \sqrt{\frac{W_s}{2\pi L p_0}} \quad ,$$

where L is the typical length of the stroke. To get W_s, they utilized the formula given by Remillard (1960) as

$$W_s = \rho L K^2 \left(\frac{di}{dt}\right)^2 / 8\pi \quad ,$$

where K is a constant given as 10^{-3} cm^2/amp . di/dt is the time rate of the stroke current whose value ranges somewhere in the neighborhood of $10^9 \sim 10^{10}$ amp/sec . They obtained value of R_0 as $0.1 \sim 1$m . These figures mean that actually no damage may be expected from the direct effect of the thunder as a blast wave except in the near vicinity of the lightning. But it should be recalled that the range, where the characteristics of the wave is different from

that of the sound wave, spreads much greater than the length R_0 (See Subsection 3.1.2). Although it is not easy to state a definite region for this, it is estimated by Thome (1962) that the distance where U/c becomes 1.1 is about 100m, for the typical lightning stroke. This can be compared with the conclusion about the range given above by Kitagawa and Kobayashi. This wide range of results leads us to believe that the phenomenon may be more appropriatly studied by taking into account the non-linear effects. It is expected that this approach will give quite different features to the situation than the simple acoustic theory.

Thome (1962) studied the problem extensively based on blast wave theory which is regarded as the simplest possible model of the phenomenon. It is noted however that the range of $y=c^2/U^2$ given in Eq. (2.10), which is important to this problem is different from the one to the general blast wave problem. In this problem we are interested in a region where y is almost 1, to which the series expansion solution is apparently inadequate, and a solution valid for a wider range is needed. For this purpose, Thome utilized the approximation solution given in Section 2.4 (Eqs. (2.58)-(2.68)). Another difficulty is that the pressure wave form, needed in making a comparison with the observational data, is to be described as a function of time by an observer at a fixed point. However, all results given in the blast wave theory are functions of x, y and are appropriate to give relation to fixed time observation, and the transformation necessary to get the relation is not straight forward. This difficulty can be avoided since we are interested only in a region where y is very close to 1 by making use of the fact that the formula can be simplified.

The solution of Eq. (2.66) near $y = 1$ is found as,

$$J \sim \frac{1}{(\gamma-1)(\alpha+1)} - \left(\frac{\gamma}{\gamma^2-1}\frac{1}{\alpha+3} + \frac{1}{\gamma^2-1}\frac{1}{\alpha+1}\right)(1-y) + O[(1-y)^2] \ .$$

With use of this expression, we get from Eq. (2.68),

$$y\left(\frac{R_0}{R}\right)^{\alpha+1} \sim \frac{2\gamma}{\gamma^2-1}\frac{1}{(\alpha+1)(\alpha+3)}(1-y) + O[(1-y)^2]$$

or $$y\left(\frac{R_0}{R}\right)^2 \sim \frac{1}{4}\frac{\gamma}{\gamma^2-1}(1-y) + O[(1-y)^2] \qquad \text{for } \alpha = 1 \ ,$$

which gives

$$R \sim \frac{1}{2} KC \left(\frac{U^2}{c^2} - 1\right)^{-1/2} \ , \tag{3.61}$$

where we have defined $KC = 4\sqrt{\frac{\gamma^2-1}{\gamma}}\, R_0$.

The approximate formula (3.61) proves to be good up to about $U/c \sim 1.1$ (or $y \sim 0.826$), when compared with the numerical solution of Eq. (2.66). This is accurate enough for the present purpose. As noticed in Section 2.4, Eq. (2.66) is not precise enough to give the exact behavior of J near $y = 1$. Therefore Eq. (3.61) is different from the exact asymptotic behavior near $y = 1$, which should be as $R^{-3/4} \propto 1-y$. Nevertheless the formula may provide some rough picture as to the characteristics of the non-linear nature of the phenomenon which differ from those features given by the sound wave theory. The procedure may be improved by using the exact asymptotic expression, which requires us to determine unknown quantities involved in the expression in some way by fitting it with the known solution. It should also be recalled that we are mainly interested in the pressure distribution, to which the approximate formula given in Eq. (2.62) is rather good in whole range of y .

Now k in Eq. (3.61) has the dimension of the time, and it turns out that K is very small compared with the time of arrival for most of the observation made at points reasonably far away from the lightning. Consider the case of $R = 0.1 - 1m$ given above. By taking $c = 340$ m/sec, $\gamma = 1.4$, k becomes $10^{-3} - 10^{-2}$ sec. (Thome (1962) used the energy value of 2.1×10^{10} joules given in Schonland (1950) and 6.0×10^{3} m for L, which gives 2×10^{-2} sec for k, which is slightly greater because of the larger value for energy). In any case, the observation time for an observer at, say, 100 meters from the lightning will be about 0.3 second which is more than ten times k. Utilizing this fact, the transformation of the pressure function given in Eq. (2.62) for the fixed time scheme, can be performed for $t \gg k$. The result neglecting small order terms in k/t, is

$$p/p_0 = 1 + \frac{1}{4} \frac{\gamma}{\gamma+1} \left(\frac{k}{t}\right)^2 \left(\frac{3r^2}{c^2 t^2} - 1\right) .$$

which gives the pressure distribution observed at a distance from the lightning stroke r as a function of the time t. It is more convenient to rewrite the equation in the following form.

$$pr^2 = p_0 r^2 + \frac{4\gamma^2}{(\gamma+1)(\gamma^2-1)} E_1 \left(\frac{r}{t}\right)^2 \left\{\frac{3}{c^2}\left(\frac{r}{t}\right)^2 - 1\right\} ,$$

where we have used the expression for k and R_0 given in Eqs. (3.61) and (2.14). This relation can be expressed in a single graph of pressure versus time for all observed distances by including r as a scaling factor on the axis. Such a curve given by Thome (1962) for $E_1 = 3.5 \times 10^{6}$ joules is reproduced in Figure 18, where r is in kilometers.

These results, as well as the frequency spectrum of the pressure wave form

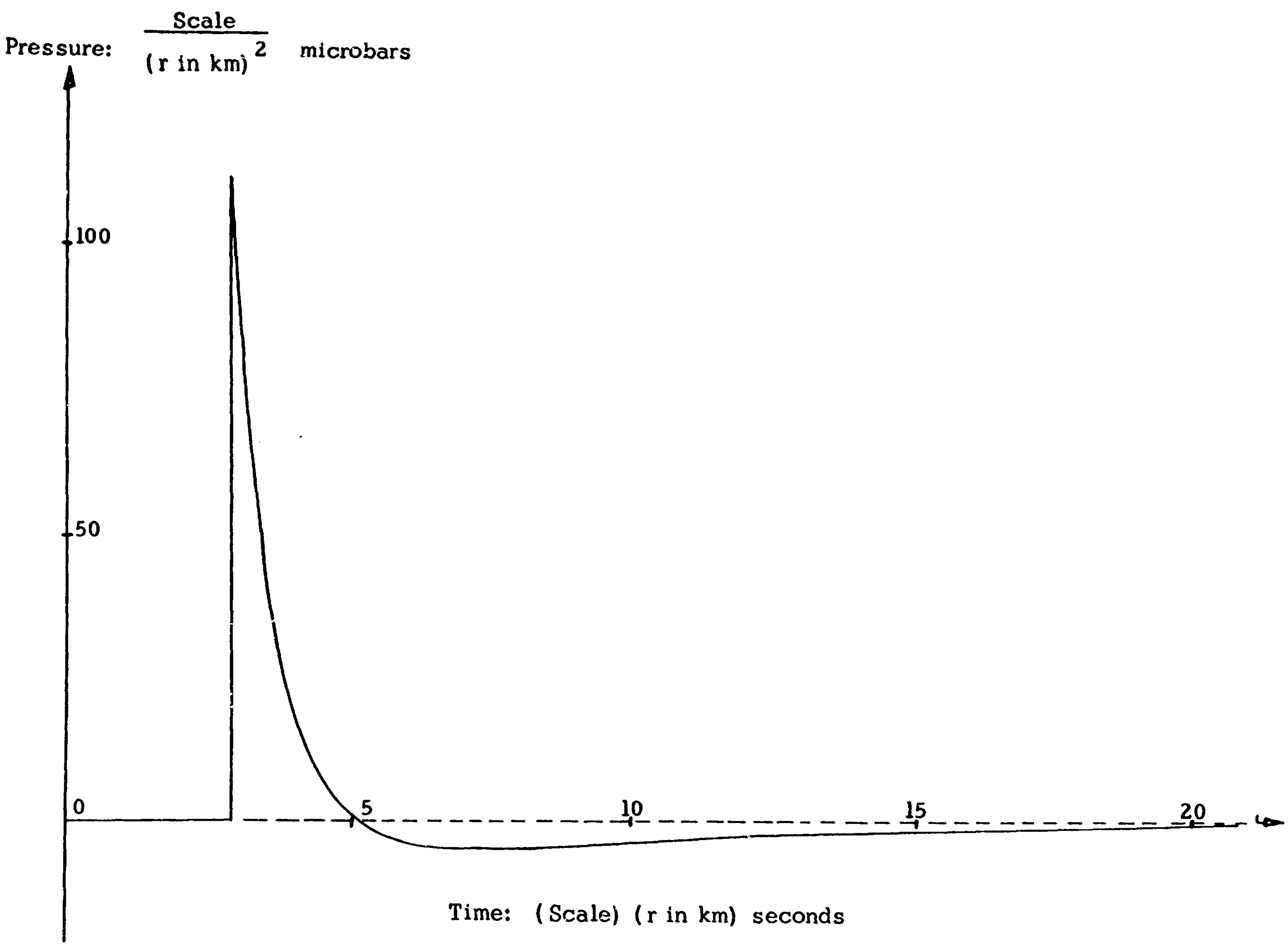
Pressure: $\frac{\text{Scale}}{(\text{r in km})^2}$ microbars
100
50
0
5
10
15
20
Time: (Scale) (r in km) seconds

Figure 18

given above were compared with observational data (which are rather rare) and also with the theoretical study based on linear theory by Remillard (1960). It was found that the results here are significantly different from those given on a linear basis, but support recent observational data by Arabadzhi (1952), while there are disagreements with earlier observations (Schmidt, 1914).

It is rather surprising that very few studies have been undertaken in the acoustics of thunder from the observational or the theoretical standpoint. It will be very interesting to see further developments in this fascinating field of study, which seems to be in its infancy.

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